This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

NYPL RESEARCH LIBRARIES
3 3433 06909640 6

A TREATISE ON PRACTICAL
PLANE AND SOLID GEOMETRY.

A TREATISE

ON

PRACTICAL

LANE AND SOLID GEOMETRY

CONTAINING

SOLUTIONS TO THE HONOURS QUESTIONS SET AT
THE EXAMINATIONS OF
THE SCIENCE AND ART DEPARTMENT (1887–1896)
INCLUSIVE.

ILLUSTRATED WITH OVER 200 DIAGRAMS.

BY

THOMAS JAY EVANS,
LECTURER ON GEOMETRY AND ARCHITECTURAL SUBJECTS AT THE SOUTH-WEST
LONDON POLYTECHNIC INSTITUTE, MANRESA ROAD, CHELSEA;

AND

W. W. F. PULLEN, Wh.Sc., A.M.I.C.E.,
PROFESSOR OF MECHANICAL ENGINEERING, SOUTH-WEST LONDON POLYTECHNIC
INSTITUTE, MANRESA ROAD, CHELSEA.

LONDON: CHAPMAN & HALL, LD.
1897.

THE NEW YORK
PUBLIC LIBRARY

ASTOR, LENOX AND
TILDEN FOUNDATIONS.
1897.

PREFACE.

DURING the last few years, the syllabus of the Honours Stage of the Annual Examination in Practical Geometry has undergone considerable change, and matter has recently appeared in the papers which bears upon work that, up to the present, has not been adequately dealt with in text-books for Honours students.

The authors have not attempted to completely fill this gap, but, in solving the problems in a number of recent examination papers, giving hints and proofs in most cases, together with the theories of Isometric Projection, Shadows, and Perspective, with a few interesting miscellaneous examples, they believe they are assisting the average student who has arrived at this stage of his work; and, at the same time, there are many of the less difficult problems which the student, who is approaching the Honours Stage, might read with advantage.

It is very probable that this work would not have been begun, had not Mr. Evans a large number of solutions previous to the conception of the idea of publishing the same; and in these he cannot forget the debt he owes to the excellent instruction he received some years previous at the hands of Mr. Henry Angel.

After much consideration it was decided that, as a student who is prepared to carry on the work at this stage should be thoroughly acquainted with the fundamental problems in Practical Geometry, such problems should be omitted from the introduction, but that ample references should be made where necessary to existing standard works. At the same time, descriptions of methods employed in the fundamental problems are often added to the text, for the sake of supplying a more complete sequence in the reasoning.

Proofs have been added where such are deemed necessary, and a series of simple kinematical exercises have been added at the end of the volume by Professor Pullen, these being seldom dealt with except in works on mechanics.

Almost throughout the work H.P. and V.P. have been used to signify the horizontal and vertical planes respectively, and other well-known contractions have been used throughout the text.

THE AUTHORS.

CONTENTS.

CHAPTER I.

PAGE
Plane Geometry 1

CHAPTER II.

Graphic Statics 60

CHAPTER III.

Solid Geometry—Loci 103

CHAPTER IV.

Solid Geometry—Lines and Planes 132

CHAPTER V.

Section of Surfaces and Solids 169

CHAPTER VI.

Isometric Projection 254

CHAPTER VII.

Shadows 270

CHAPTER VIII.

Perspective 302

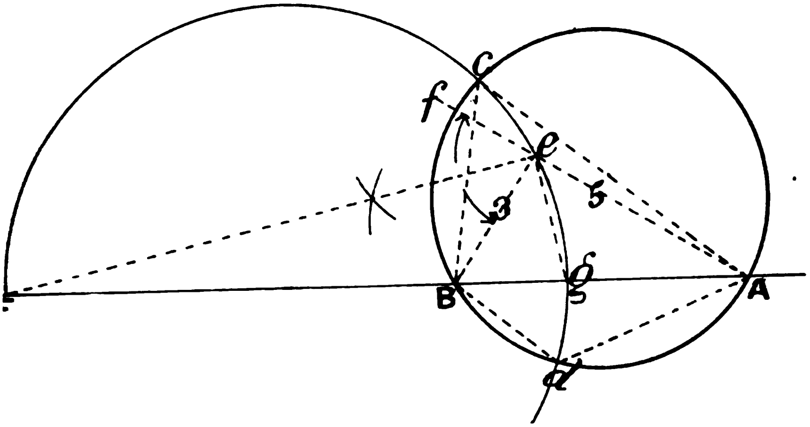
CHAPTER IX.

Miscellaneous Examples 338

GEOMETRY.

CHAPTER I.

PLANE GEOMETRY.



5, **Find a third point C on the circumference of the circle such that CA : CB : : 5 : 3.**

On AB construct any triangle having its sides as 5 : 3; such as △ B*e*A; produce A*e* to *f*, and bisect ∠ B*ef*. With *e*F meeting AB produced in F, draw *eg* perpendicular to *e*F, meeting AB in *g*. On F*g* as diameter, describe a circle passing through the given circle in points *c* and *d*. Then *c* and *d* are the points required; for the circle *dgec*F is the locus of apices of all triangles constructed on base AB having its sides in the ratio of 5 : 3.* (NOTE.—The line AF is said to be harmonically divided.†)

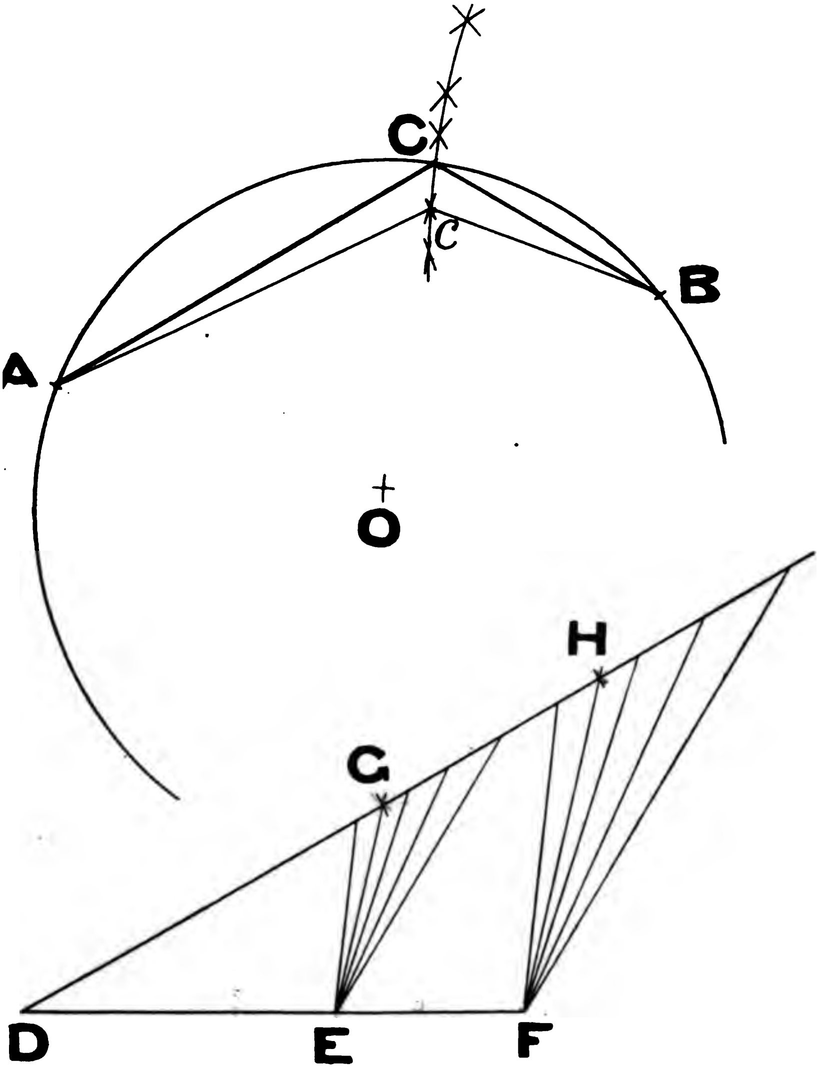
---

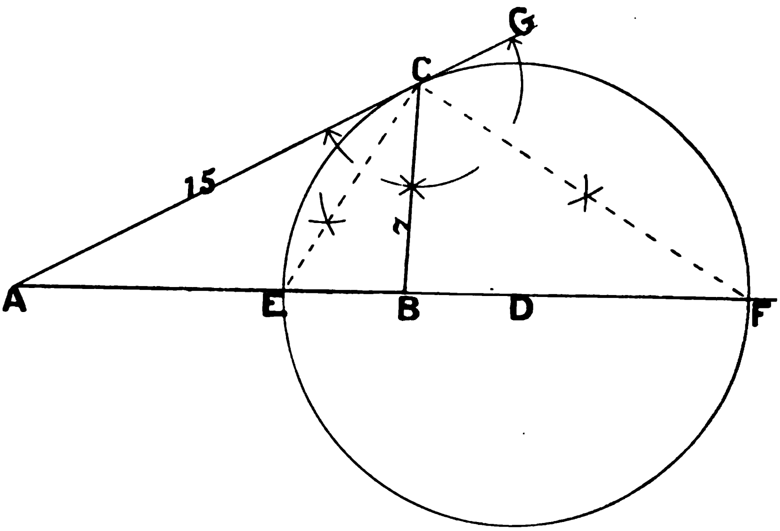
* Casey's "Sequel to Euclid," p. 71.

† *Vide* Eagle's "Plain Curves," Prob. 17.

No. 51, 1893. *Second Method* :—

Find the locus of points whose distances from A and B are in the ratio of 5 : 3; thus, draw any line DE and produce it to F, making DE : EF : : 5 : 3. Draw a line DGH, making any angle with DEF. From D along DH take any length, say DG, and join GE. Through F draw FH parallel to GE, cutting DG in H. Then DG : GH : : DE : EF : : 5 : 3. With A as centre and DG as radius describe an arc; and with B as centre and GH as radius describe another arc, cutting the first arc in *c*. Therefore *c* is a point on the locus. In the same way get other points on the locus and join them up, forming a continuous curve, which cuts the circle in C. Join AC and BC. Then AC : CB : : 5 : 3.


C
C
B
A
O
H
G
D
E
F

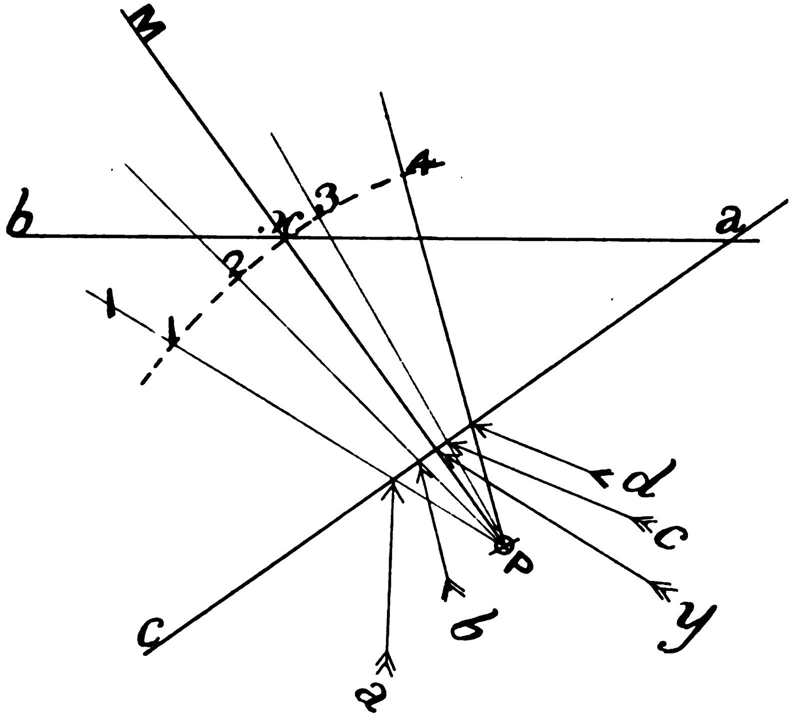


No. 42, 1889. **A line AB is $3\frac{1}{4}$ in. long. Draw the locus of the vertex of the triangle described on it as base, the sides of which are in a given ratio (7 : 15).**

Draw a line AB $3\frac{1}{4}$ in. long, and construct on it any triangle whose sides are in the ratio 7 : 15, vertex C. Bisect ∠ ACB, and produce the bisector to meet AB in E. Next from C draw CF perpendicular to CE, cutting AB produced in F (note line CF bisects ∠ BCG). On the line EF describe a semicircle ECF; then ECF is the locus of the vertex of the triangle required, whose sides are in the ratio 7 : 15.* The line AB is said to be harmonically divided in E and F, and C (AEBF) is a harmonic pencil.

---

* *Vide* Eagle's "Plain Curves," Prob. 17, p. 21; also Casey's "Sequel to Euclid," p. 71, Prob. 7.



Given two lines *ab*, *ac* **and a point** *p*. **Draw through** *p* **a line cutting the given lines in such a way that the intercepted portion is 1⅛ in. (a curve may be employed in the solution).**

From point *p* draw a series of lines as *p*1 . . . 4, cutting line *ac* in points *abcd*; cut off lengths *a*1, *b*2, *c*3, *d*4 each equal 1½ in: Draw a curve passing through these points, cutting line *ab* in point *x*. Draw line *p*M through the point *x*, and the intercepted portion *yx* equals 1½ in.

No. 53, 1893. **Determine the locus of the point A, such that when joined to B, C, D, and E, the sum of the areas of the triangles ABC and ADE shall equal 1 square inch.**

Let A be any point on the locus, and let AM and AN be perpendiculars let fall from A on to DE and BC respectively.

The area of the two triangles $= \frac{1}{2}(\text{BC}.\text{NA} + \text{DE}.\text{AM})$

$= 1$ sq. in.

Divide both sides of the equation by $\frac{\text{ED}}{2}$;

$$\text{Then } \frac{2}{\text{DE}} = \frac{\text{BC}}{\text{DE}}.\text{NA} + \text{AM}$$

As NA and MA are the only variables in the equation, it is of the form—

$$y = mx + c$$

which is the equation to a straight line; hence the locus of the common vertex A is a straight line. It is only necessary to find two points on this line.

In the above equation—

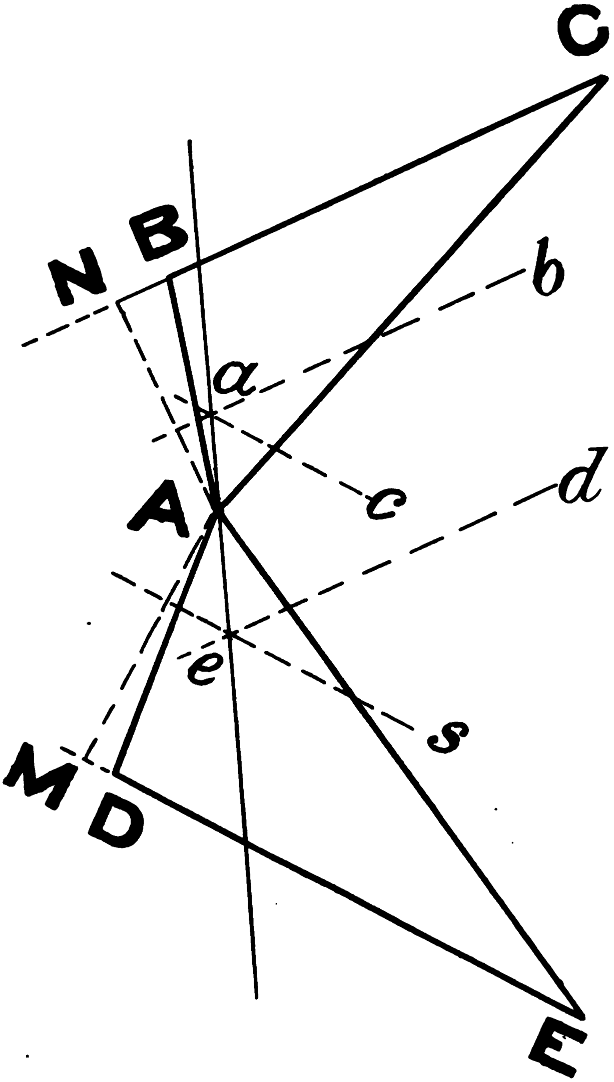
$$\text{when NA} = 1, \text{ AM} = \frac{2 - \text{BC}}{\text{DE}}$$

$$\text{and when AM} = 1, \text{ NA} = \frac{2 - \text{DE}}{\text{BC}}$$

In the original figure BC = 1·3 in., and DE = 1·47 in. Putting in these values, we find that AM = 0·475 in., and NA = 0·405 in.

Draw *ed* parallel to BC and 1 in. away from it. Similarly, draw *es* parallel to DE and 0·475 in. away from it. The intersection of these lines gives the point *e* on the locus.

Draw *ab* parallel to BC and 0·405 in. away from it, and draw *ac* parallel to DE and 1 in. away from it. The intersection of these lines gives the point *a* on the locus. Join *ae*. This is the required locus.


C
B
N
b
a
c
A
d
e
s
M
D
E

Second Method :—

No. 53, 1893. **Find any point A such that, when joined to B, C, D, and E, the sum of the areas of the triangles ABC and ADE shall be equal to one square inch. Determine the locus of all points, such as A, which fulfil that condition.**

The lines BC, DE are given. Let BC, DE intersect in O; then cut off OD = DE, and OB_1 = BC. Then B_1D_1 * will be the direction of the locus of point A.

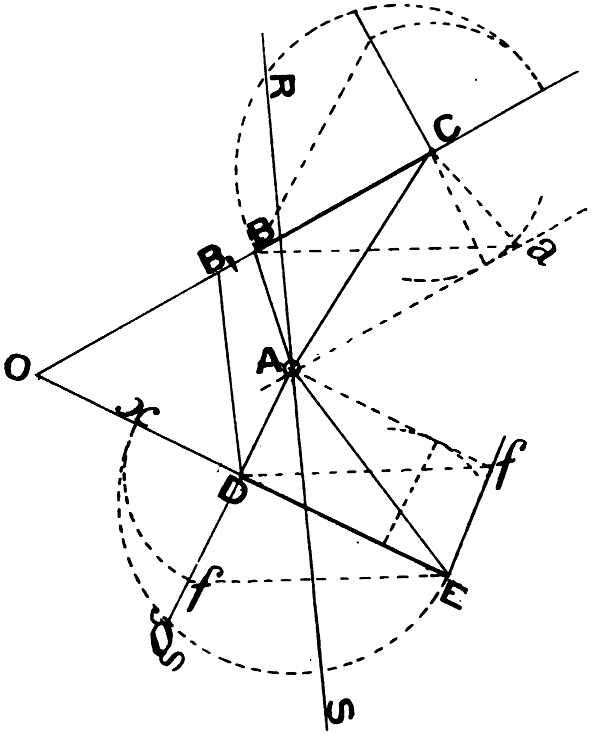
Now on DE and BC construct a triangle of $\frac{1}{2}$ sq. in. area, such as DE*f*, BC*a*; † then through *a* draw a line parallel to BC as *a*A, and line *f*A parallel to DE, meeting in point A: then A is the point required for the sum of the areas of triangles BCA, and DEA = 1 sq. in. Through A draw the locus RS parallel to B_1D_1. No further description of this is necessary, as all the construction lines are left.

NOTE.—The method given on page 6 should not be used in a pure geometry examination. It is chiefly analytical. For a shorter method of solving this problem than either of those just given, see Miscellaneous Examples.

---

* See Casey's "Sequel to Euclid," Prob. 8, p. 6.

† On DE construct a rectangle of 1 sq. in. area, thus: Make D*g* = 1 in.; describe a semicircle to pass through *g*, having its centre on DE, and meeting DE produced in *x*: then D*x* is the required side of rectangle of 1 sq. in. area. Draw the line *f*E; then the triangle D*f*E = $\frac{1}{2}$ rectangle DE, D*f* = $\frac{1}{2}$ sq. in. area.


R
C
B
B₁
a
O
A
x
f
D
f
E
g
S

No. 51, 1896. **C_1, C_2 are the centres of two circles of radii 1 in. and $\frac{5}{8}$ in. respectively. Find, *by a geometrical construction*, the locus of points such that the difference of the squares of tangents drawn to the two circles from any point on the locus be equal to 1 sq. in.**

Analysis.—Let p be any point on the locus, and pm, pn the tangents to the circles. Rotate the triangle pmc_1 about c_1 to the position hac_1; also the triangle pnc_2 to dc_2b. Then $ha = pm$, and $pn = db$. Draw any line $osut$, and ow perpendicular to it and equal in length to 1 in. If $ou = pn$ and $uw = pm$, then $wu^2 - ou^2 = wo^2 = 1^2$ (Euc. I. 47).

Construction.—Mark off points s, u, t, etc., along ot; then set up ah perpendicular to c_1c_2 and touching the circle. If ah is to be a tangent to the circle, then the point h will always be at a distance hc_1 from c_1; therefore with c_1 as centre and c_1h as radius describe the arc hp. Similarly, the end of the tangent db must lie somewhere on the circle pd; therefore p is a point on the locus. Similarly, q and r are points on the locus. The locus is found to be a straight line.

Second Method (figure, p. 13):—

Let p be a point on the locus, and de the radical axis. The radical axis of a pair of circles is the locus of points from which *equal* tangents can be drawn to the two circles. Also let o be the middle point of c_1, c_2.

Now, by Euc. I. 47—

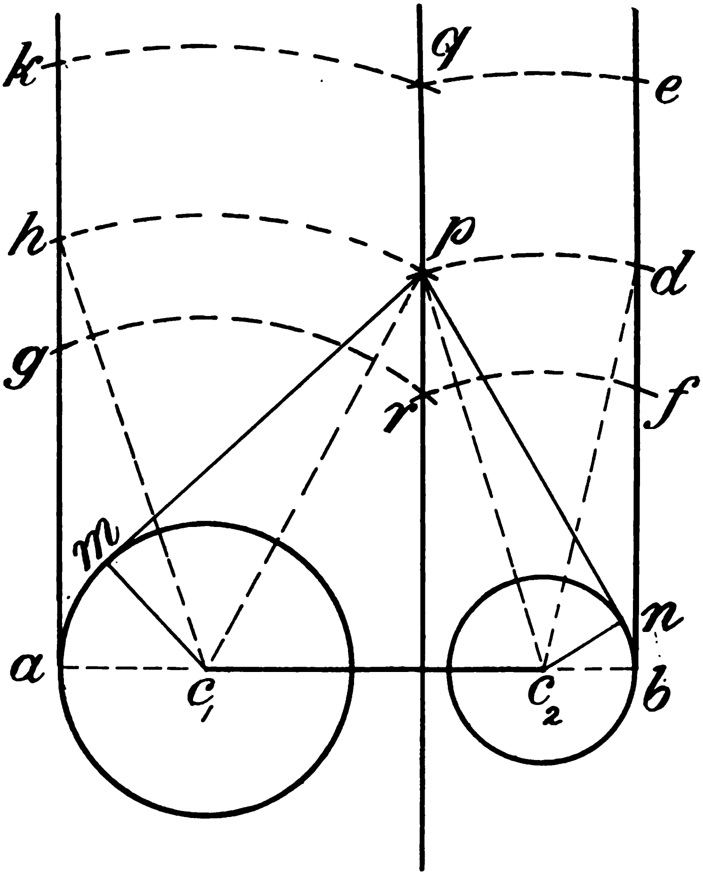
$$c_1p^2 = c_1g^2 + gp^2$$

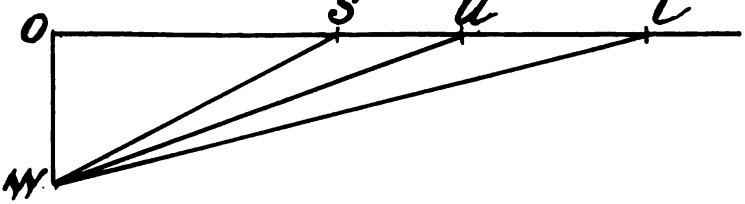
also—

$$c_2p^2 = c_2g^2 + gp^2$$

Subtract the latter from the former, and we get—

$$\begin{aligned} c_1p^2 - c_2p^2 &= c_1g^2 - c_2g^2 \\ &= (c_1g + c_2g)(c_1g - c_2g) \end{aligned}$$


k
q
e
h
p
d
g
r
f
m
n
a
c_1
c_2
b


o
s
u
t
w

Now—

$$c_1g + c_2g = c_1c_2$$

also—

$$c_1g - c_2g = c_1o + og - (oc_2 - og) = 2og$$

therefore after substitution we get—

$$c_1p^2 - c_2p^2 = c_1c_2 \times 2og \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

From *d*, a point on the radical axis, equal tangents, *dx* and *dy*, have been drawn to the two circles; therefore, by Euc. I. 47—

$$c_1d^2 = c_1x^2 + xd^2$$

also—

$$c_2d^2 = c_2y^2 + dy^2$$

Subtract the latter from the former; remembering that $dx = dy$, and we have—

$$c_1d^2 - c_2d^2 = c_1x^2 - c_2y^2 = c_1f_1^2 - c_2f_2^2 \quad . \quad \text{(ii.)}$$

because they are radii of the same circles.

It has been proved in equation (i.) that, if *p* be *any* point in space, and c_1c_2 a line in space, *o* being its middle point and *g* the foot of the perpendicular from *p* on to c_1c_2, the difference of the squares of the distances of *p* from the extremities of the line c_1c_2 equals the length of the given line c_1c_2 multiplied by twice the distance from the middle of the line to the foot of the perpendicular from *p*.

Now, *d* is also a point in space; therefore, by the same reasoning, we get—

$$c_1d^2 - c_2d^2 = c_1c_2 \times 2od \quad . \quad . \quad . \quad \text{(iii.)}$$

Combining equations (ii.) and (iii.) we have—

$$c_1f_1^2 - c_2f_2^2 = c_1d^2 - c_2d^2 = 2 \,.\, od \,.\, c_1c_2 \quad . \quad \text{(iv.)}$$

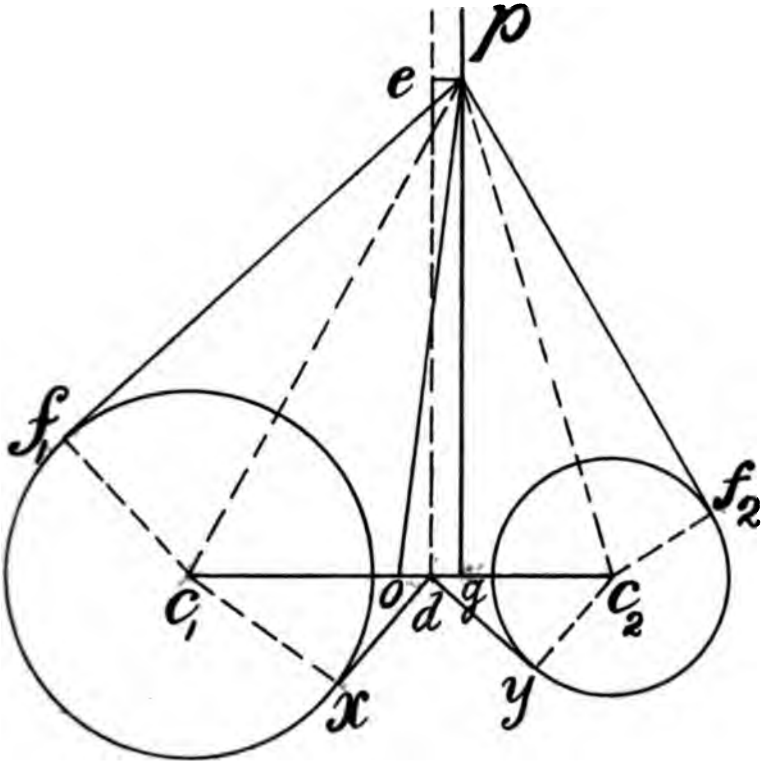
Subtract equation (iv.) from equation (i.), and we get—

$$(c_1p^2 - c_1f_1^2) - (c_2p^2 - c_2f_2)^2 = 2c_1c_2(og - od)$$

But—

$$c_1p^2 - c_1f_1^2 = pf_1^2, \text{ and } c_2p^2 - c_2f_2^2 = pf_2^2$$

therefore the last equation becomes, after substituting—

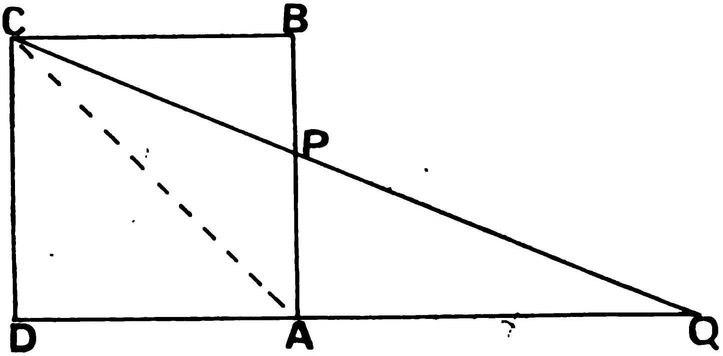


$$pf_1^2 - pf_2^2 = 2 . c_1c_2 . dg$$
$$= 2c_1c_2 . ep$$

Now, c_1c_2 is constant, and by hypothesis $pf_1^2 - pf_2^2$ is constant, namely one square inch; therefore *ep* is constant, and equal to—

$$\frac{1}{2c_1c_2} = \frac{1}{2 \times 2{\cdot}25} = 0{\cdot}22 \text{ inch}$$

Hence, set off from *d* a distance $dg = {\cdot}22$ inch, and draw a line perpendicular to c_1c_2. This is the locus of the point required.



No. 41, 1889. **Draw a square ABCD of 2 in. side (the angular points being taken in order), and through C draw a line meeting AB in P and DA produced in Q, so that the area of the triangle PAQ shall be double that of the triangle PBC.**

Construct the square ABCD of 2 in. side; join CA. Now $CA^2 = CB^2 + BA^2$ (Euc. I. 47).

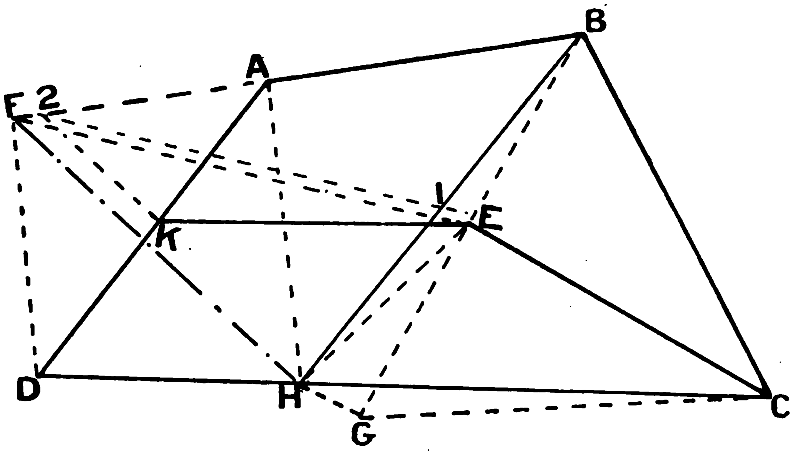
But $CB^2 = BA^2$, being sides of a square. $\therefore CB^2 = \frac{1}{2}CA^2$.

Now produce DA to Q, making AQ = CA. Join CQ, cutting AB in P. Then P shall be the point required.

Because AQ = AC (constr.), $\therefore AQ^2 = AC^2$. But $CB^2 = \frac{1}{2}AC^2$; $\therefore CB^2 = \frac{1}{2}AQ^2$.

In triangles CBP, APQ, ∠ CBP = ∠ PAQ, being right angles. And ∠ BPC = ∠ APQ (Euc. I. 15); ∴ ∠ BCP = ∠ AQP.

And triangles CBP, APQ are similar triangles. And since the square on the base BC of △ CBP is half the square on the base AQ, ∴ △ CBP is half the area of △ APQ; or △ APQ is double the area of △ CBP, because the areas of similar figures are to one another as the squares on their homologous sides (Euc. VI. 19).



41, 7. **ABCD is a field; CE is a hedge. How must the hedge be carried on in a straight line so as to divide the field into two equal parts?**

Let ABCD be the given figure, CE the given hedge.

Commence by joining E to B, forming the △ CEB. On the opposite side of CE construct a △ CEG equal in area to the △ CEB and reduce it to the △ CEH; then the portion of the field enclosed by the △ CEH equals the portion enclosed by the △ CEB. What is now required to be done is to draw from the point E a line EK dividing the five-sided figure HEBAD into two equal areas. To do this, reduce it into a four-sided figure of equal area, such as BEHF;* join EF, and bisect the diagonal HB in point 1. Draw 1 2 parallel to EF, cutting side BF in point 2. Next draw 2K parallel to AE, cutting the side AD of the original figure in point K. Then EK will bisect the remaining portion, and is the hedge required. For the line E2 bisects the quadrilateral BEHF (see one of the early miscellaneous examples in the last chapter), therefore the △ BE2* is half the area of the figure BEHDA. The △ AEB is common to the △ BE2 and the half portion BEKA. Taking this away from both, we have left the △ E2A and △ EAK. These are on the same base EA and between the same parallels EA and 2K, therefore they are equal to one another, and consequently the line EK bisects the area BEHDA.

---

* Thus, draw FD parallel to AH; join HF; then the △ AHF = △ AHD (Euc. I. 37). Hence the quadrilateral ABEF = five-sided figure ABEHD.

* The lines E2 and AE are omitted in the figure.

NOTE.—For a shorter method of solving this problem, see Miscellaneous Examples.

No. 42, 1888. **A and O are given points; *bc* is a given straight line. Draw from O a line O*d* which shall meet *bc* in a point *i*, such that the angles *aio*, *cid* are equal.**

With O as centre, radius O*b*, describe circle *abk*, which is tangential to *bc* at point *b*; through *a* draw a tangent *ai* to the circle, cutting *bc* in *i*; draw *oi*, *oa*, *ob*. Because the radii *oa* and *ob* are equal, and the side *oi* is common to the two triangles *oai* and *obi*, also the angles *oai* and *obi* are right angles; therefore the remaining angles *oia* and *oib* are equal (Euc. I. 4). Also *oib* = *cid* (Euc. I. 15). Therefore the ∠ *aio* = the ∠ *cid*.

NOTE.—Should the point *a* fall inside the circle, the solution is impossible.

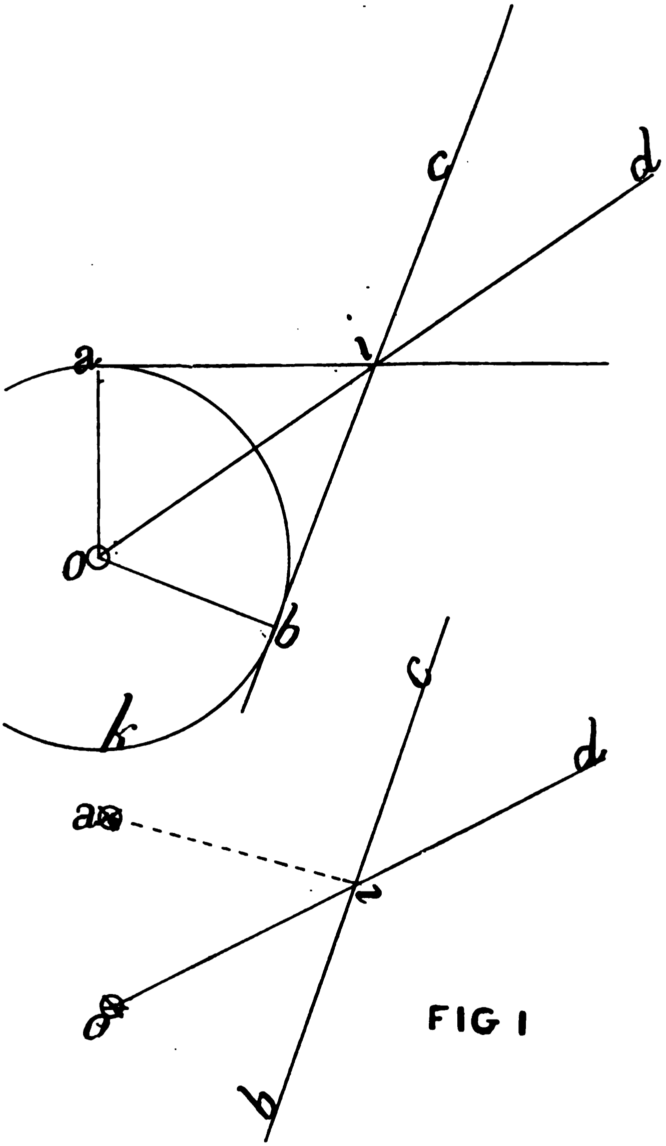


FIG I

No. 48, 1889. **A and B, two given points, are 2 in. apart. A line is drawn through each, and these lines are turned round A and B with equal angular velocities. Draw the locus of their intersections—**

1. When they revolve in the same direction.

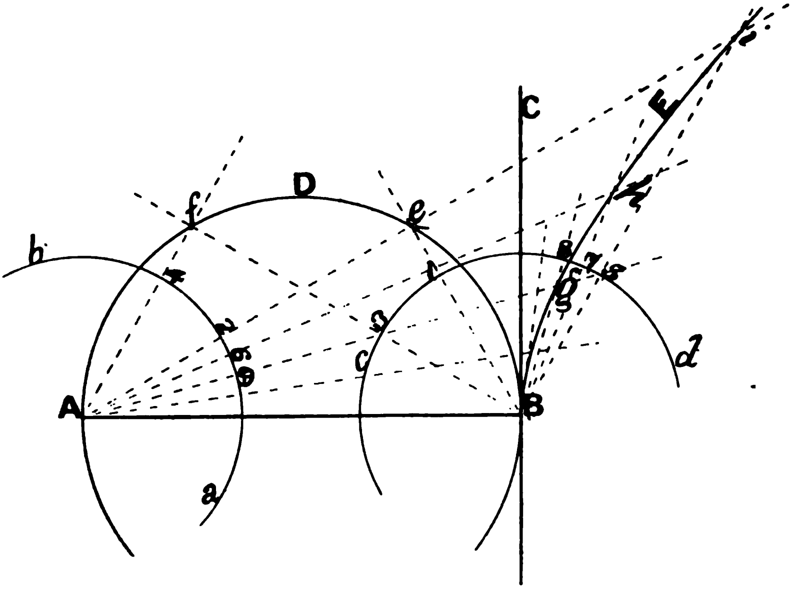
2. „ „ „ opposite direction.

The initial position of lines to be taken, one coinciding with AB, and the other perpendicular to it.

Let A and B be the points, and the initial position of the lines to be AB, BC.

1. With A as centre, any radius, describe the arc *a*24*b*; and with B as centre, and the same radius, describe the arc *c*31*d*. Suppose the line AB to have turned to the position A2, and the line BC to the position B1, the ∠ BA2 = ∠ CB1. Produce A2, B1 to cut each other in point *e*; then *e* is one point on the curve of intersection. Again, let AB have turned to A4, and B6 to B3—these lines intersect in *f*; then *f* is another point on the curve; and so on a series of points may be obtained. But it is not necessary to find points in this manner, for A and B are at right angles in their initial position, and as their angular velocities are the same, the angle at their intersection will be a right angle, as A*e*B, A*f*B; therefore on AB describe a semicircle A*fe*B, which will be the locus of their intersection.

2. As before, let A, B be the points, AB, BC the given lines. Now they turn in opposite directions, therefore let AB turn to A2, and BC turn to A5, where ∠ BA2 = CB5. Produce A2 to meet B5 in *i*; then *i* is a point on the curve. In the same way find a number of points, such as the intersection of A6 and B7 in *h*, A9 and B8 in *g*. The ∠ BA6 = ∠ CB7, and ∠ BA9 = CB8. Through the points so found draw the curve *ihg*B, which is the second locus of the intersection of the lines. This curve will be a hyperbolic curve.

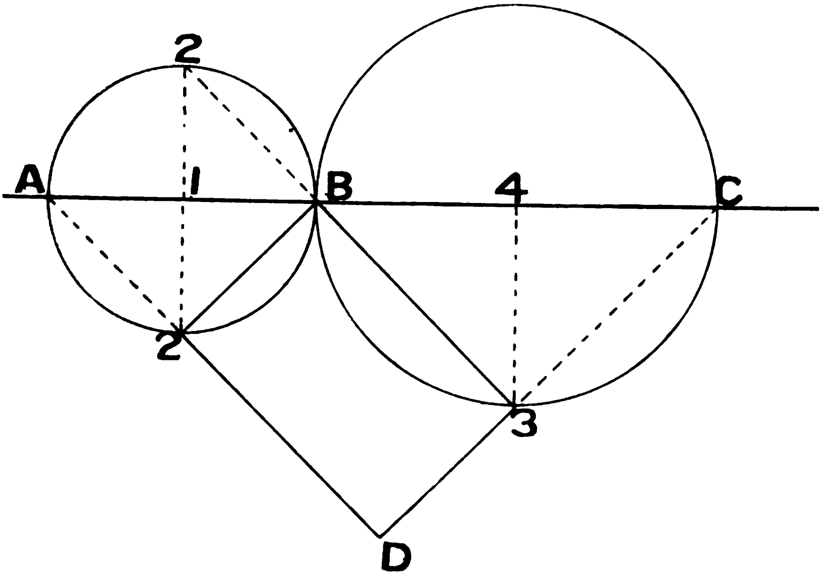

C
D
E
f
e
h
b
4
2
6
9
3
t
c
A
B
a
8
7
5
g
d
i

No. 51, 1891. **Take three points A, B, C in a straight line; AB = $1\frac{1}{2}$ in. BC = $2\frac{1}{4}$ in., B between A and C. Draw a rectangle, the sides of which are in the ratio 2 : 3, vertex at B, and two sides passing respectively through A and C.**

On AB describe the circle A2B, and on BC describe circle B3C; in the circle AB draw the radius 1 2. Then the angle contained by the chords A2, 2B is a right angle; also the angle contained by the chords B3, 3C in circle BC is a right angle; and as the required rectangle is to have its sides to pass through A and B, having its vertex at B, then B2, B3 will be two sides of the required rectangle, and A2 and C3 produced meeting in D, will be the other two sides. Because the angles A2B, B3C in circles AB, BC are right angles, then A2, 2B and B3, C3 will be the direction of the sides, having their ratio as 2 : 3, because the diameters AB : BC : : 2 : 3;

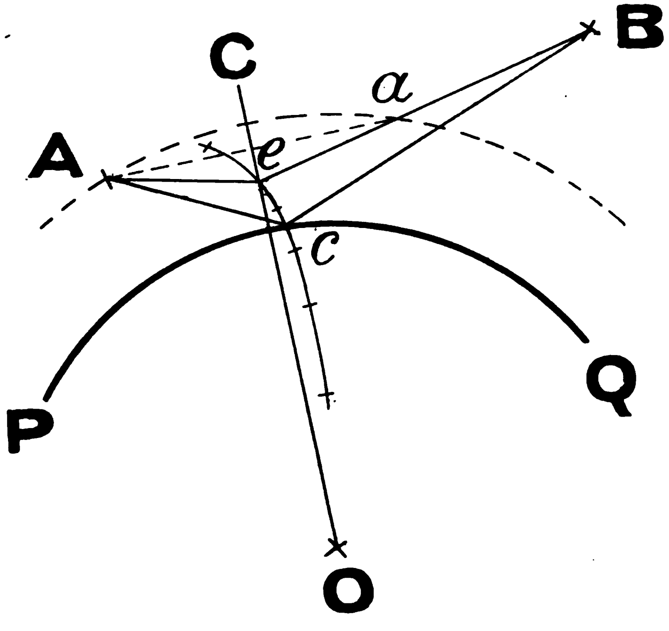
Also AB : BC : : A2 : 2D
and A2 = 2B,
∴ 2 : 3 : : AB : BC : : B2 : 2D

Therefore the rectangle 2B3D fulfils the required condition.


2
A
I
B
4
C
2
3
D

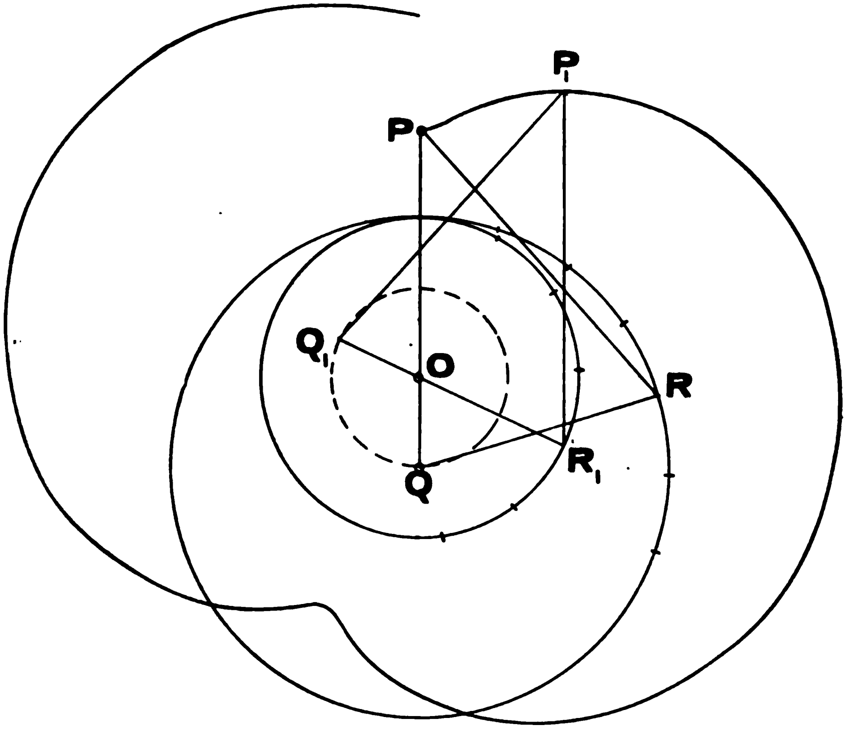
No. 52, 1896. **PQ is a portion of the circumference of a circle of which O is the centre; A and B are two given points. Find,** ***by a geometrical construction,*** **a point *c* on the circumference such that lines drawn from A and B to *c* shall make equal angles with the radius O*c*. (A geometrical curve, such as a conic, may be employed to determine the required point.)**

Take any radius OC of the given circle PQ. The virtual image of A in OC is at *a*, as far on the other side of OC as A is in front of it. Join B*a*, and produce it to cut OC in *e*; join A*e*: then the angle A*e*C = the angle B*e*C. Repeat the construction with different radii of the circle PQ, and other points similar to *e* will be obtained. These points can be joined up to form the continuous curve shown in the figure, which cuts the circle PQ in *c*. The property of the curve *ec* is that if any point be taken in it, and that point joined to A and B, then these lines will make equal angles with the line joining the point to the centre O; and consequently the point *c* where the curve cuts the circle PQ is the required point, and A*c* and B*c* make equal angles with *c*O. This is the same as saying that a ray of light emitted at A will be reflected at the point c to B. It will be noted that this method is extremely accurate, as the curve *ec* cuts the circle PQ nearly at right angles. Also the images of A will all lie on the dotted circle through A.


B
C
a
A
e
c
P
Q
O

No. 52, 1894. **A circle of 2¼ in. diameter rolls on a fixed circle of 1¾ in. diameter, the smaller circle being *inside* the larger one. Draw the curve traced by a point moving with the larger circle, and 1⅞ in. from its centre, during one complete revolution.**

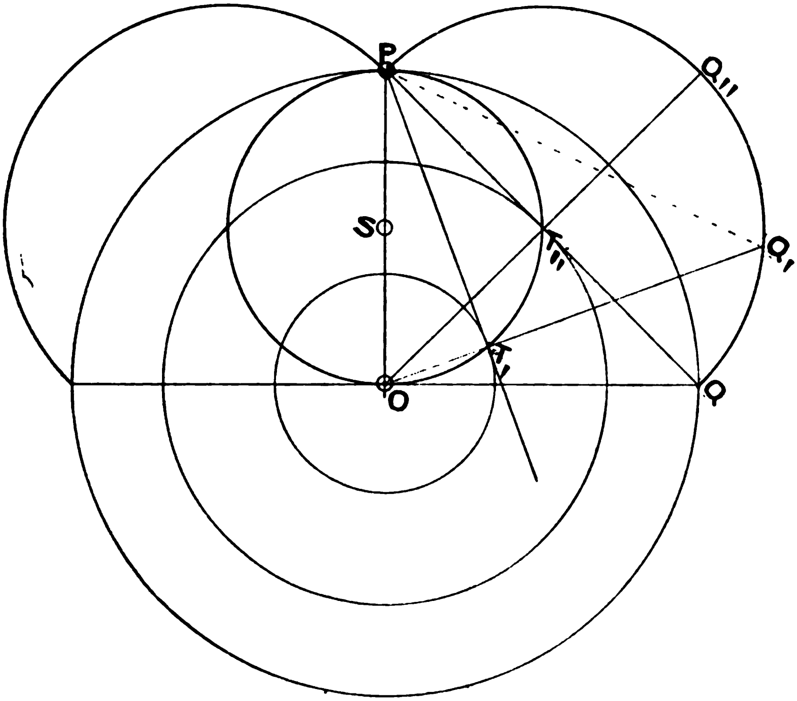
With centre O describe the circle passing through R_1 with radius ⅞ in. Then describe another circle, whose radius is 1⅛ in., to touch the first circle, as shown. The centre of this second circle will be at Q. Set off QP = 1⅞ in. Divide off the circumference of the smaller circle into some number of equal parts, and step off from the point of contact round the large circle the same equal parts; (and it is convenient to number the divisions in each circle, beginning from the point of contact.) The point R is the fourth division on the large circle, while R_1 is the fourth on the small circle. These two points will come into contact as the large circle moves round. When in contact, the original line QR will now be Q_1R_1. The point Q will move round O in a circle (shown dotted) whose radius equals the difference in radii of the two circles through R and R_1. When R coincides with R_1, QP will be at Q_1P_1, and PR comes to P_1R_1. The triangle PQR is the same as $P_1Q_1R_1$. Repeat this construction for every division point, and the locus of P is the curve PP_1.


P₁
P
Q₁
O
R
R₁
Q

o. 51, 890. **From a point P tangents are drawn to a series of concentric circles with centre O($OP = 2\frac{1}{2}$ in.). If T_1 is the point of contact of one of these tangents, and Q_1 a point on the normal at T_1, such that $T_1Q_1 = T_1P$, draw the complete locus of point Q_1.**

Draw line $OP = 2\frac{1}{2}$ in. With O as centre describe a series of concentric circles, as T_1, T_{11}; then draw the tangents PT_{11}, PT_1, and the normals OT_{11}, OT_1 at the contact points T_1T_{11}. Now the tangent PT_1 and radius OT_1 meet at T_1, and the $\angle PT_1O$ is a right angle; then the locus of point T is a semicircle described on OP.* Now cut off a length T_1Q_1 equal to T_1P, and join Q_1P; then the angle OQ_1P is 45°. Then the locus of point Q is a segment of a circle which contains an angle of 45°; therefore, with centre T_{11}, radius TP'', describe the arc $PQ_{11}Q_1Q$, to meet a line drawn through O perpendicular to OP in point Q, where the locus ends. Similarly, the locus on the left is determined. If necessary, a series of circles may be described with centre O, and tangents and normals drawn as OT_1 OT_{11} PT_1 PT'_{11}, and the curve drawn through the points found.

* Euc. III. 31.


P
Q,,
S
T,,
Q,
T,
O
Q

No. 53, 1891. **The bar *ab* revolves uniformly about the point *a* as a fixed centre; a link, *bd*, is pivoted to *ab* at *b*, and a fixed pin at the point *c* of the link is constrained to move over a certain curve. Determine the form of this curve when, in one revolution of *ab*, the point *d* moves down to line *ab* and returns to *d* with uniform velocity in the direction perpendicular to *ab*. Trace also the complete locus of *d*.**

With *a* as centre, radius *ab*, describe circle *b*6 9; this circle is therefore the locus of point *b* in all positions during one revolution. Divide the circle into any number of equal parts, say twelve; also divide the perpendicular distance *d* to *ab* into six equal parts, as 12 1, 1 2, 2 3, etc. Now suppose *ab* to have revolved to *a*1, then *d* will have moved to *d*1; also when *ab* has moved to *a*2, *d* will have moved to d_2, and so on for each of the divisions. When *d* will have reached line *ab* at point 6, it will commence to rise again to *d*7, *d*8, *d*9, *d*10, *d*11, and *d*12, its original position. A curve drawn through these points will be the locus of *d*.

The locus of point *c* is obtained in exactly the same manner; *i.e.* when *ab* will have moved to *a*1, and *bd* to *bd*,′ point *c* will have moved to c_1, and so on. The curve drawn through these points will be the locus of point C.

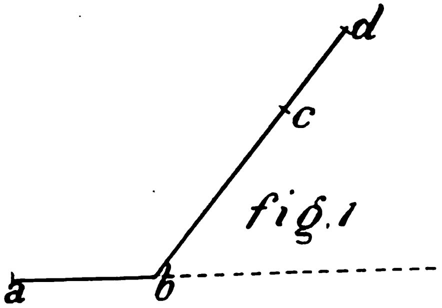
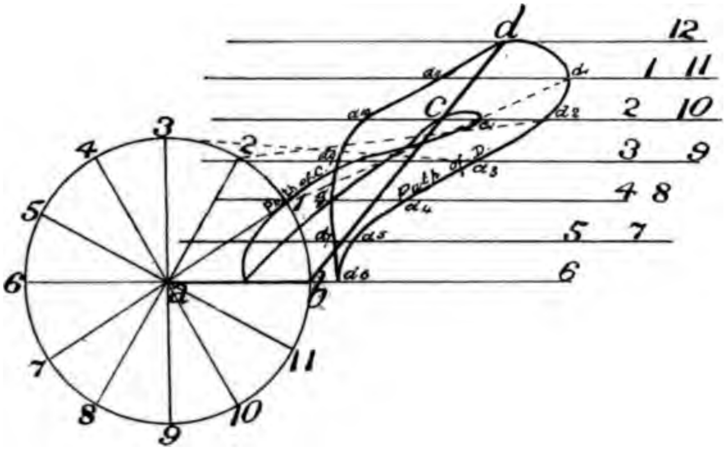
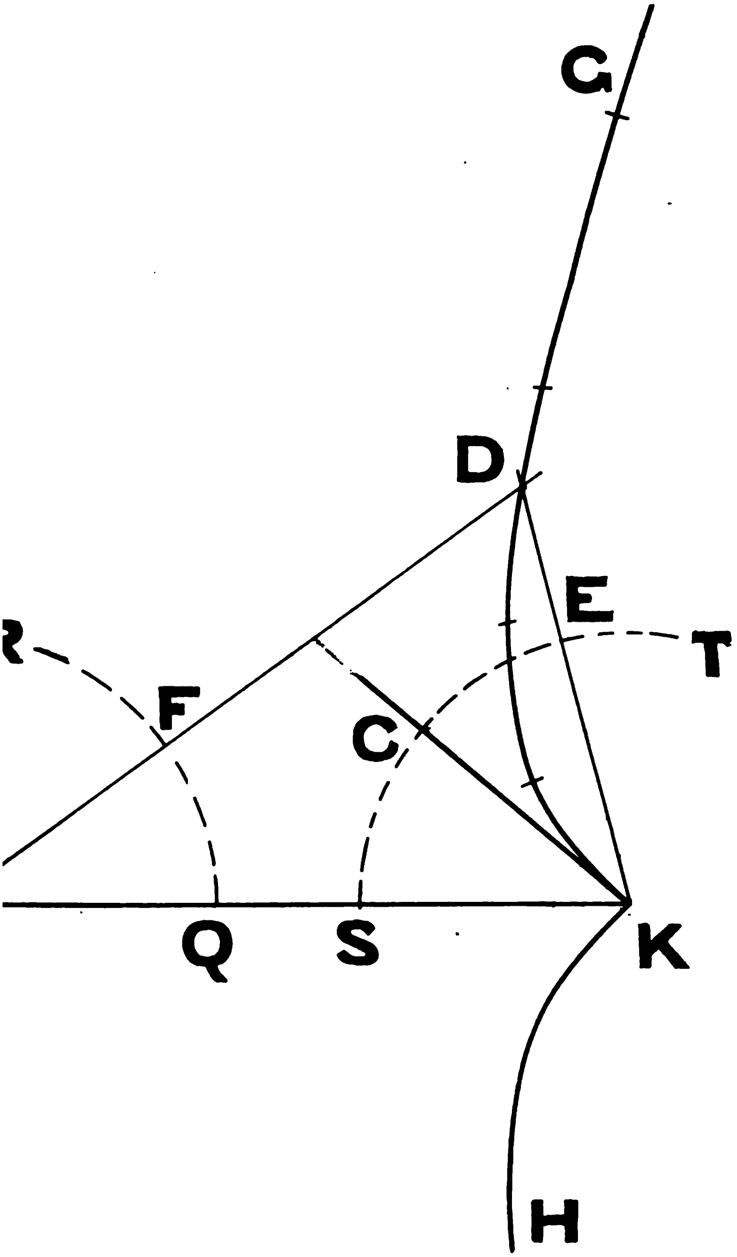

d
c
a
b

fig.1


d
12
d1
1 11
d
C
d2
2 10
d3
3 9
Path of C
Path of D
d3
4 8
d4
5 7
d5
d6
6
3
4
2
5
6
a
b
7
8
9
10
11
c
f
g

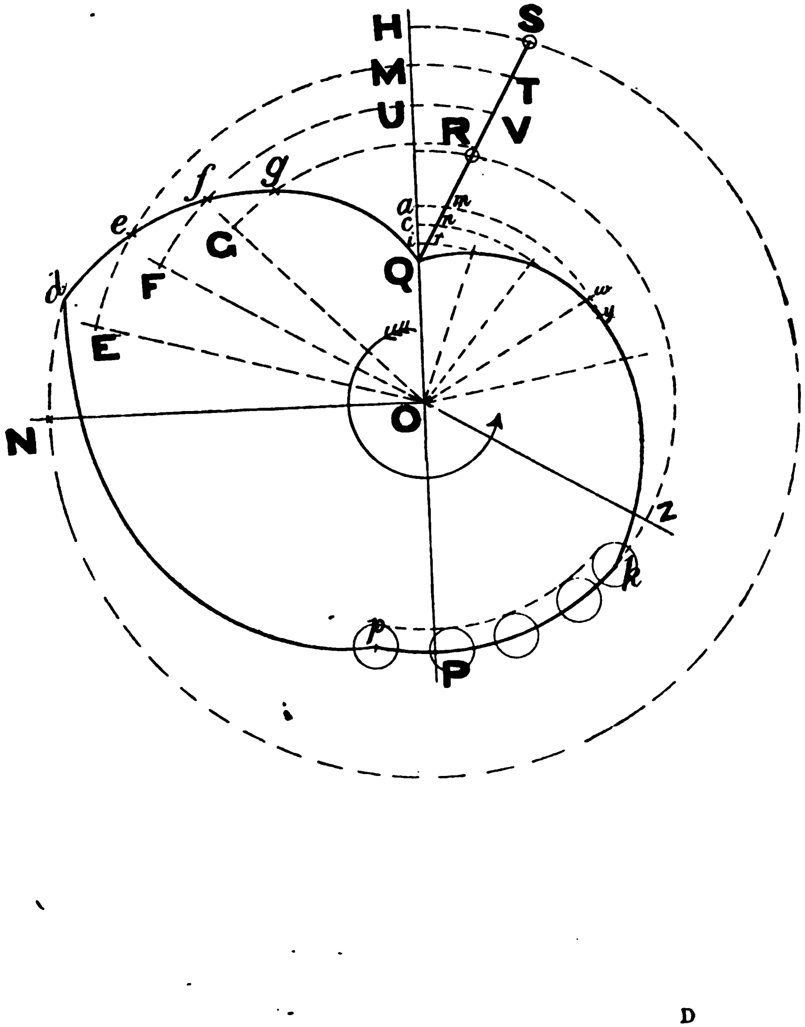
No. 51, 1895. **The base of a triangle is 2″ in length, the difference between the angles adjacent to the base is 40°. Find the locus of the vertex of the triangle. (The construction of the locus should be clearly indicated, as no marks will be given for a locus found by trial only.)**

Set down the line PK 2 in. long, and make the angle PKC equal to 40°. With centres P and K describe arcs RQ and ST with the same radius. From C along the arc CT mark off any length CE, and from Q mark off an equal length QF. Join PF and KE, and produce them until they meet in D. Then D is the vertex of a triangle whose angles DKP, DPK at the base differ by the angle SKC = 40°. Repeat the construction a number of times, and join up the vertices thus found, and the curve KDG is obtained, which is the required locus. There will be a similar curve KH below the base-line PK. Also the curve GDK will be continued beyond K, when DK comes on the lower side of CK, and PD moves down below PK. This portion of the curve is not shown in the figure. The curve is a hyperbola.


G
D
E
T
F
C
Q
S
K
H

No. 43, 1887. **Obtain the form of cam plate to be keyed on to a shaft whose centre is O, and impart motion to a point Q, which travels in the given line QRS. In one-third of a revolution of the cam shaft Q is to be raised at uniform speed to R. It is then to remain at rest for one-sixth of a revolution, and after that to rise to S at a uniform rate in one-quarter of a revolution, and then fall back from S to Q in one-quarter of a revolution.**

Set off OZ such that the angle QOZ = 120° = one-third of a revolution. Divide QOZ into six equal parts; the dividing lines are shown dotted. Also divide QR into six equal parts, and with O as centre draw circles (dotted) which shall pass through the dividing points in QR. At the end of the third interval O*w* will have moved to OQ, and the point *y* will be in contact with the point *m* in QR, therefore set off *wy* = *am*; then *y* is a point on the cam. From *k* to *p* the curve will be a circle, because the point R will be stationary during that interval. The curve *pd* will be described in the same manner as Q*k*, also the curve *defgQ*; but for the latter SQ must be divided into six equal parts. If the moving point in QR had been a roller, then, instead of the present cam outline, we should have had a parallel outline at a distance away equal to the radius of roller, as shown dotted between *p* and *k*.


H
S
M
T
U
R
V
g
f
e
a
m
c
n
i
r
G
Q
d
F
w
y
E
N
O
Z
k
p
P

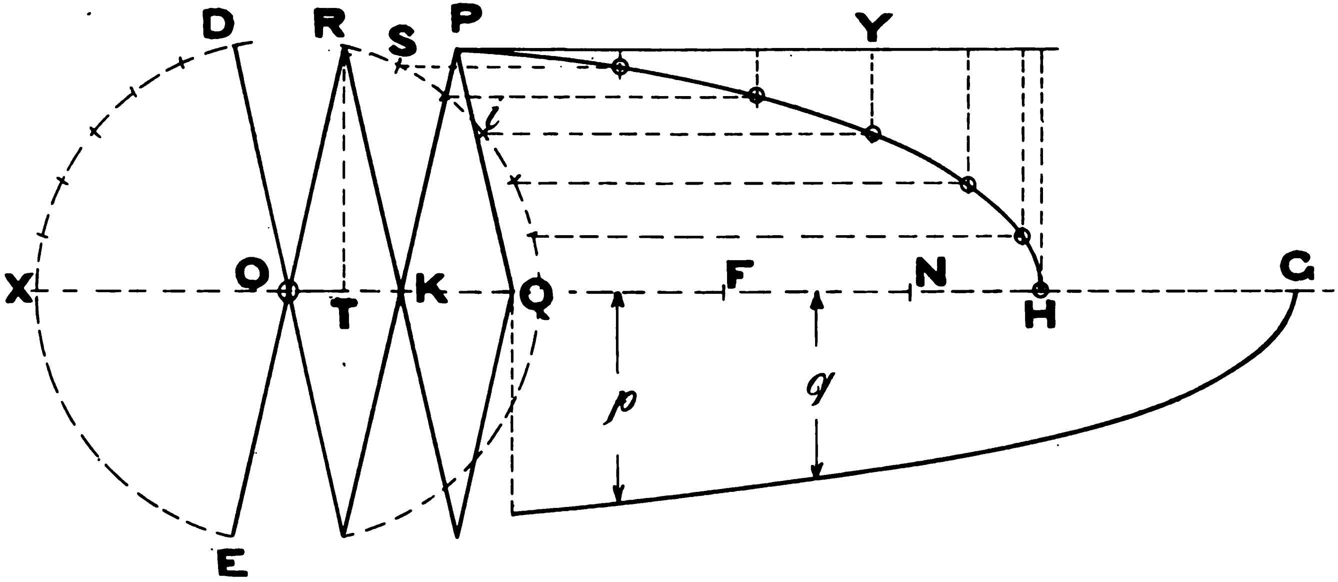
No. 53, 1894. **The given figure represents an arrangement of jointed bars forming "lazy tongs." Assuming the point *o* to remain fixed, and the arms OD, OE to be brought into coincidence on OX moving with a uniform angular velocity—**

(a) Draw the curve traced by the point P.

(b) Draw a curve representing the velocity with which the point Q moves.

Divide off the arc RQ into any number of equal parts, in this case six. ER being one continuous rod, the whole of it will rotate about O with the same angular velocity at any particular instant, consequently the angular velocity of OR is constant when that of OE is constant. From the nature of the mechanism, the joint K between O and Q will move horizontally twice the distance that R will move horizontally. The point P will move in the same time a distance = movement of K + the horizontal movement of R = three times the horizontal movement of R, *i.e.* three times the horizontal distance between S and RT. Set off along PY a distance = three times the horizontal distance between S and RT. The level of the new position of R will be that of S, and hence the first new position of P is found. Similarly, the third position of P will be at the intersection of the vertical through Y with the horizontal through *i*, PY being three times the horizontal distance between *i* and RT. Then PH is the path traced out by P.

Velocity is the distance moved over per unit of time. Take as the unit of time that in which R takes to move from R to S. As all the intervals are of the same magnitude, the distance moved by Q during each of these intervals will be a measure of the velocity of Q. While R is moving to S, Q will move to F. At the middle point of QF erect a perpendicular p = QF. Repeat the construction for every interval, and the curve passing through the extremities of the perpendiculars will be the velocity curve. It will be noted that the point Q moves horizontally four times the horizontal distance traversed by the point R.


D
R
S
P
Y
t
X
O
T
K
Q
F
N
H
G
p
q
E

No. 52, 1890. **A conical vessel in the form of a right cone (height 4in., radius of base $1\frac{1}{4}$in.), vertex down, is filled with liquid, which is allowed to flow out of the vertex at a uniform rate. Draw a curve representing the rate at which the surface of the liquid falls.**

Let the water-level at any time t be that shown in the figure at a height h above the vertex. The radius of the surface we can denote by r, and that of the base of the cone by R, while the total height of cone = 4 in. = H, say.

The volume of a cone is $\frac{1}{3}$ × height × area of base $= \frac{\pi}{3}hr^2$

But from similar triangles $\frac{R}{H} = \frac{r}{h}$, or $r = \frac{R}{H}h$

And volume of cone of water $= \frac{\pi}{3} \cdot \frac{R}{H}h^3 = V$, say.

The rate at which the water flows out is constant, *i.e.* the decrease of volume per unit of time is constant, and therefore $\frac{dV}{dt} =$ constant.

Differentiate the above equation with respect to t, and we get a constant $= \frac{dV}{dt} = \frac{2\pi}{3} \cdot \frac{R}{H} \cdot h^2 \cdot \frac{dh}{dt}$.

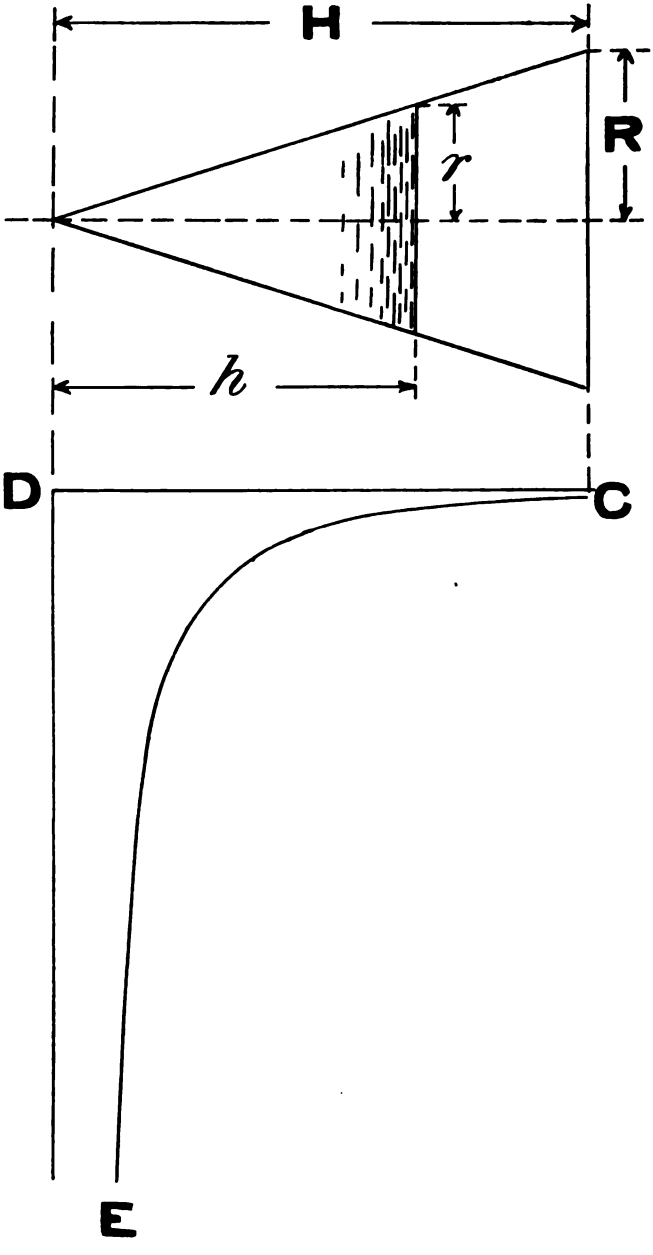
Now $\frac{dh}{dt} =$ rate at which surface falls per unit of time; therefore $\frac{dh}{dt} = \frac{\text{constant}}{h^2}$.

Let the constant be unity; then—

when $h = 0,\ \frac{dh}{dt} = \infty$ when $h = 1,\ \frac{dh}{dt} = 1$

" $h = \frac{1}{4},\ \frac{dh}{dt} = 16$ " $h = 2,\ \frac{dh}{dt} = \frac{1}{4}$

" $h = \frac{1}{2},\ \frac{dh}{dt} = 4$ " $h = 3,\ \frac{dh}{dt} = \frac{1}{9}$


H
R
r
h
D
C
E

$$\text{and when } h = 4,\ \frac{dh}{dt} = \frac{1}{16}$$

Take a base-line CD parallel to the axis of cone, and plot the numbers given in the table thus found. Then EC is the required curve.

This curve has been obtained by pure differentiation after the volume of the cone of water has been found. Analytical differentiation would hardly be admitted in a Pure Geometry examination, although the process would be carried out under the cloak of the geometrical interpretation of finite differences, such as is used in the miscellaneous examples relating to velocity. Thus: draw a curve on a line parallel to the axis of cone as base, whose ordinates represent volumes of cones of water of different heights. As the volume of a cone of water is given above as a constant × h^3, the curve will be a cubic. It is given in the question that $\frac{dV}{dt}$ is constant; therefore—

$$\text{constant} = \frac{dV}{dt} = \frac{dV}{dt} \cdot dh$$

$$\text{and hence } \frac{dh}{dt} = \text{a constant} \times \frac{dh}{dV}$$

Select dV and the constant both equal to unity; then draw a curve whose ordinates represent the corresponding dh. The ordinates of this curve will give the rate at which the surface falls.

No. 52, 1895. **The quadrilateral *abcd* is given, also the four points P′, Q′, R′, S′ (upper part of the figure). Draw through these four points a quadrilateral, having its sides proportional and similarly placed, and its angles equal and similarly placed, with those of *abcd*. The sides proportional to *ab*, *bc*, *cd*, and *de*, to pass respectively through P′, S′, R′, and Q′.**

The points P′, S′, R′, Q′ have been transferred to P, S, R, Q in the middle of the figure. Join RQ, QP, and PS. The side proportional to *ad* must pass through Q, and the side proportional to *dc* must pass through R; therefore QR is the chord of a circle whose segment, QDR, will contain an angle, QDR, equal to the angle *adc*. Hence, the locus of D is a circle.′ Describe on QR a segment of a circle which shall contain an angle equal to *adc* = 40°.

Similarly, the locus of A is the circle QAP, whose segment, QAP, contains an angle equal to *dab* = 120°. In the same way the locus of B is the circle PBS, whose segment, PBS, contains an angle, PBS, equal to *abc* = 100°.

The intersection of the first two circles is at M, and of the second two circles at N.

Join MQ and MR; the point D is not yet found, but it may be assumed as found in the analysis. The same applies to A and B. If one of them were known, the problem would be solved; hence we must proceed to find one of them—say the point A.

Join MD; then, because the angles QRM and QDM are in the same segment, they are equal to one another, and QRM has been found. Similarly, the ∠ QAM = ∠ QPM, both being in the same segment, and the angle QPM is known. Therefore in the △ ADM the angles at A and D, and consequently that at M, are known. Draw the △ A′D′M′, similar to ADM; thus, set down any length A′D′, say, 4 in., and at A′ and D′ make angles equal to QPM and QRM respectively, then D′M′ and A′M′ are 2·86 and 1·9 in. long respectively.

Again, the ∠ PAN = ∠ PMN, which is known; and in the same way the ∠ PBN = ∠ PSN, which is known, and therefore the angles of the △ ABN are known.

Set down A″B″ of any length, say 4 in., and at A″ and B″ make angles equal to PMN and PSN respectively; then the △ A″B″N″ is similar to ABN, and B″N″ = 3·82 in. also A″N″ = 4·13 in.

Now, from similar triangles—

$$\frac{\text{DA}}{\text{AM}} = \frac{\text{D}'\text{A}'}{\text{A}'\text{M}'} = \frac{4}{1{\cdot}9}$$

$$\text{therefore AM} = \text{AD} \times \frac{1{\cdot}9}{4}$$

$$\text{Similarly, } \frac{\mathrm{AN}}{\mathrm{AB}} = \frac{\mathrm{A''N''}}{\mathrm{A''B''}} = \frac{4{\cdot}13}{4}$$

$$\text{hence AN} = \mathrm{AB} \times \frac{4{\cdot}13}{4}$$

And after dividing one result by the other, we have—

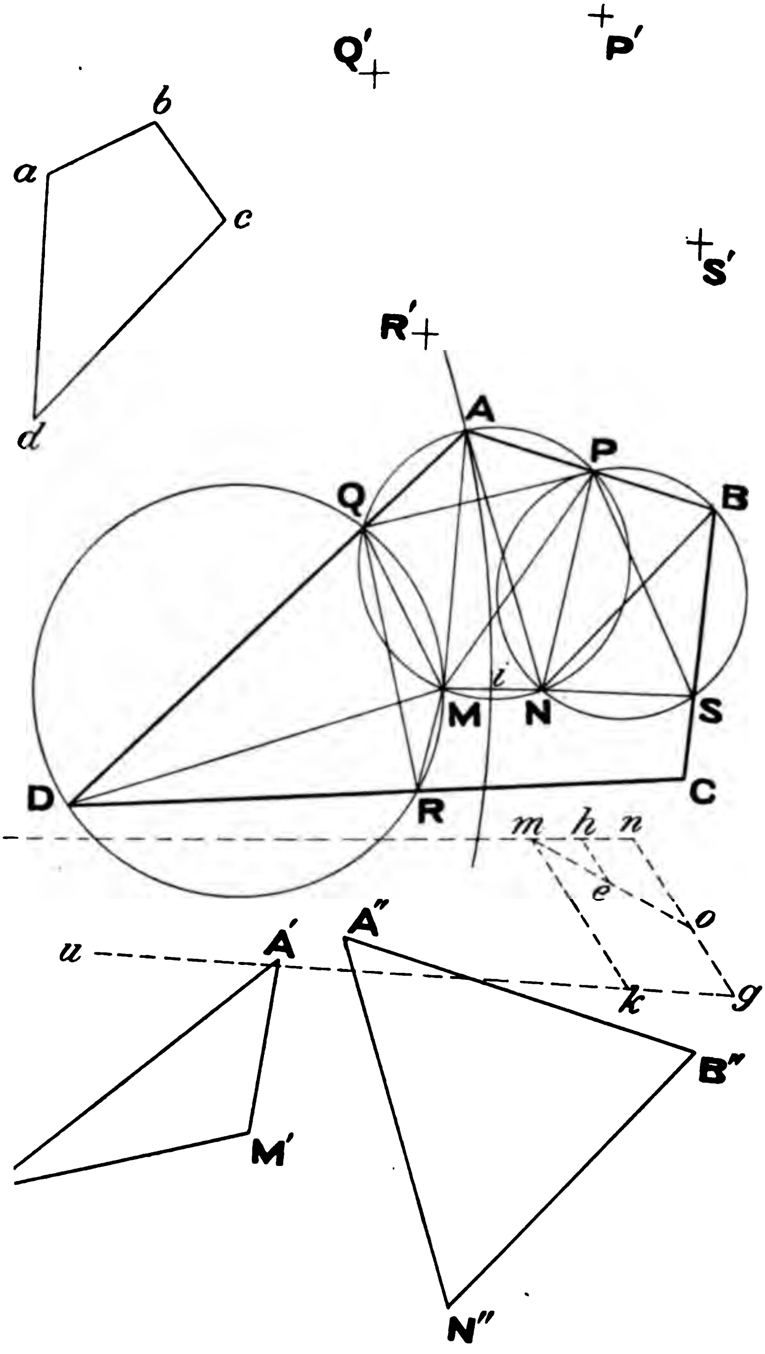
$$\frac{\mathrm{AM}}{\mathrm{AN}} = \frac{\mathrm{AD}}{\mathrm{AB}} \times \frac{\frac{1{\cdot}9}{4}}{\frac{4{\cdot}13}{4}}$$

$$\text{But } \frac{\mathrm{AD}}{\mathrm{AB}} = \frac{ad}{ab} \text{ by hypothesis} = \frac{2{\cdot}63}{1{\cdot}26}$$

$$\text{therefore } \frac{\mathrm{AM}}{\mathrm{AN}} = \frac{2{\cdot}63}{1{\cdot}26} \times \frac{1{\cdot}9}{4{\cdot}13} = \frac{0{\cdot}96}{1}$$

Hence the problem is now reduced to drawing two lines from M and N respectively in the ratio of 0·96 to 1, and meeting in the circle QAP. (This is the problem No. 51 of 1893.) For the sake of clearness, this part of the problem has been separated out from the general figure, and is shown dotted. Set off *mm* = MN. It is required to divide *mn* internally and externally in the ratio of 0·96 to 1. Draw any line *mo*, making *me* = 0·96, and *eo* = 1. Join *on*, and through *e* draw *eh* parallel to *on*. Then *h* is the point of internal division. Next, set up any line *ng* to represent 1 (this is to a larger scale than *eo*), and parallel to it make *mk* to represent 0·96. Join *gk*, and produce the line to meet *nm* in *u*. The point *u* is off the paper. Then *mn* is divided externally in *u*, so that *mu* : *un* : : 0·96 : 1. On *uh* as diameter describe a circle. This circle is the locus of A such that MA : AN : : 0·96 : 1. The point *i* in MN is the same as *h* in *mn*. The intersection of this locus and the circle QAP is the position of A. Join AQ, and produce it to cut the circle QDR in D. Join AP, and produce it to cut the circle PBS in B. Similarly, draw BS and DR intersecting in C. Then ABCD is the required quadrilateral.

In connection with this problem, it will be found useful to read through problems I. to IV. in the Miscellaneous Examples.


Q'
P'
b
a
c
S'
d
R'
A
P
Q
B
i
M
N
S
D
C
R
m
h
n
e
o
A''
u
A'
k
g
B''
M'
N''

No. 41, 1888. **Divide an ellipse whose axes are 4 in. and $2\frac{3}{4}$ in. into three parts equal in area and perimeter.**

Draw the major and minor axes at right angles to each other, as AA′ = 4 in., and BB′ = $2\frac{3}{4}$ in. Draw the ellipse by trammelling (various other methods may be adopted); that is, take a narrow slip of paper and mark off a length equal to half major axis, as $x'y$, and also a length xy equal to half minor axis, then xx' equals the difference in length of major and minor axes. Now place this slip of paper in different positions, always keeping points xx' on major and minor axes respectively, and mark a series of points as at y. All these points will be on the curve of the ellipse.

Next divide the major and minor axes into six equal parts; each part will be semi-axis of the required elliptic division. Then on A1 construct a semi-ellipse, having one of such parts as semi-axis; thus $af = \frac{1}{6}$ of major axis, $fg = \frac{1}{6}$ minor axis. Now on the opposite end of the major axis, with 1A′ as major axis and $2k$ as semi-minor axis, describe the semi-ellipse $1k$A′, and repeat the above working for the next division, which will divide the ellipse into three parts of equal area and perimeter. The continuous curve Ag1$1k$A′ is equal in length to the curve ABA′, and the figure enclosed by these lines is equal to $\frac{1}{3}$ the area of the largest ellipse.

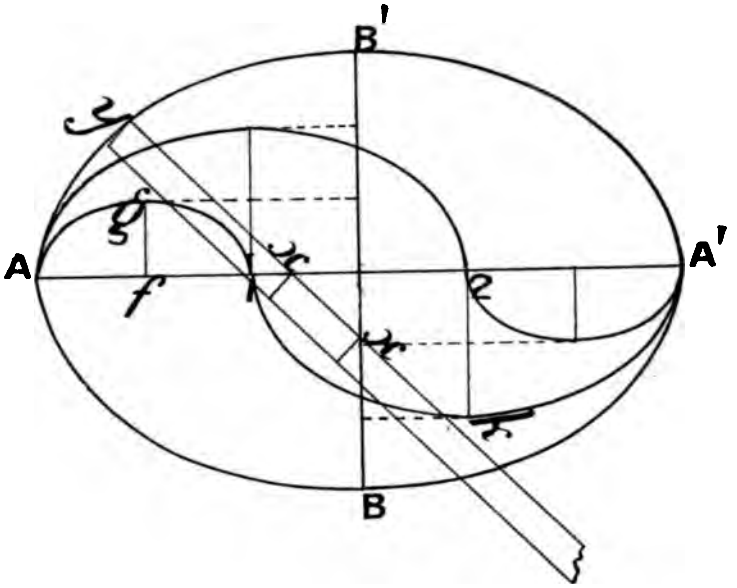
Proof.—A circle can be projected orthogonally into an ellipse, therefore the quantities in the circle will bear a constant ratio to those in the projected ellipse.

Draw any circle AGD on AD as diameter, and divide the diameter in the point C in any ratio (second figure).

$$\text{Let } \frac{\text{AC}}{\text{AD}} = n\text{; then CD} = \text{AD} - \text{AC} = \text{AD}(1 - n).$$

$$\text{Area of semicircle AGD} = \tfrac{1}{2} \cdot \frac{\pi}{4}\text{AD}^2$$

$$\text{,, \quad ,, \quad AFC} = \tfrac{1}{2} \cdot \frac{\pi}{4}\text{AC}^2$$



$$\text{Area of semicircle CMD} = \tfrac{1}{2}\cdot\frac{\pi}{4}\text{CD}^2$$

$$\begin{aligned}\text{Area AGDMCF} &= \text{area AGD} + \text{area CMD} - \text{area AFC}\\ &= \tfrac{1}{2}\cdot\frac{\pi}{4}\text{AD}^2 + \tfrac{1}{2}\cdot\frac{\pi}{4}\text{CD}^2 - \tfrac{1}{2}\cdot\frac{\pi}{4}\text{AC}^2\\ &= \frac{\pi}{8}(\text{AD}^2 + \text{CD}^2 - \text{AC}^2)\\ &= \frac{\pi}{8}[\text{AD}^2 + \text{AD}^2(1-n)^2 - n^2\text{AD}^2]\\ &= \frac{\pi}{8}\text{AD}^2(1 + 1 - 2n + n^2 - n^2)\\ &= \frac{\pi}{8}\text{AD}^2(2 - 2n) = \frac{\pi}{4}\text{AD}^2(1-n)\end{aligned}$$

Let AC be two-thirds of AD; then $n = \frac{2}{3}$, and $1 - n = \frac{1}{3}$.

Therefore, from above, the area AGDMCF $= \frac{1}{3} \cdot \frac{\pi}{4}\text{AD}^2 = \frac{1}{3}$ area of circle AGDE.

In the same way, the length of arc AGD $= \frac{1}{2}\pi \,.\, \text{AD}$.

Also length of arc AFC $= \frac{1}{2}\pi \,.\, \text{AC}$; and the length of arc CMD $= \frac{1}{2}\pi\text{CD}$.

Then length of curve AFCMD $= \frac{1}{2}\pi \,.\, \text{AC} + \frac{1}{2}\pi \,.\, \text{CD}$

$$= \frac{\pi}{2}(\text{AC} + \text{CD}) = \frac{\pi}{2}\text{AD}$$

$$= \text{length of arc AGD}$$

Therefore the length of perimeter of each third part is the same as that of the original circle.

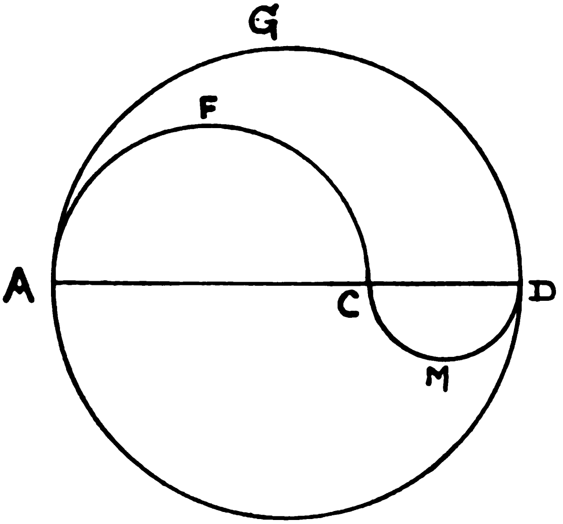
No. 51, 1892. **Determine the parabola having F as focus, and passing through P and Q.**

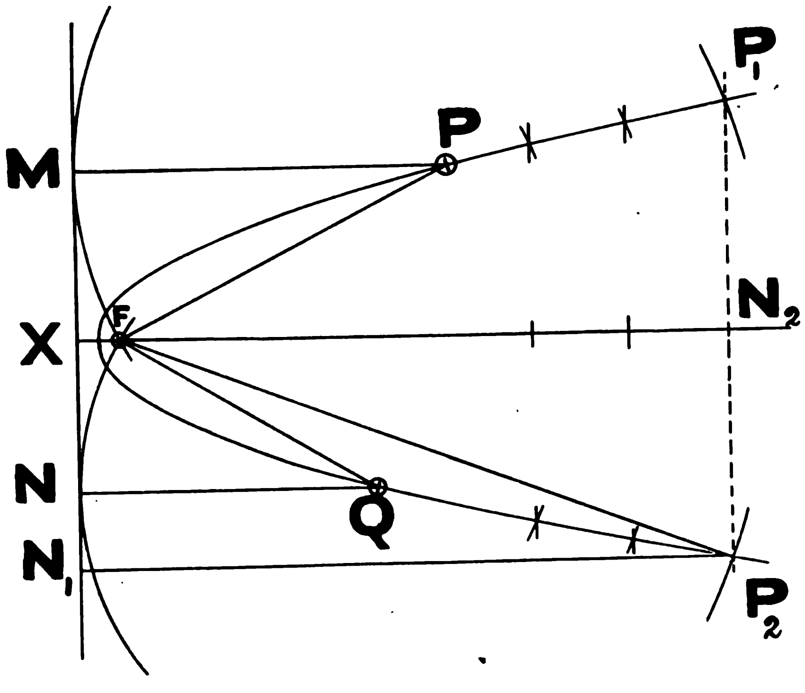
Analysis.—Let MN be the directrix; then, from the property of the parabola,* that the focal distance of any point on the curve equals the perpendicular distance of the point from the directrix, PF = PM, and QF = QN. Also as PM and QN are perpendicular to the directrix, the directrix must touch circles drawn from P and Q as centres, with radii PM and QN respectively.

Construction.—With centres P and Q draw circles whose radii are PF and FQ respectively, and then draw MN to touch these circles. Through F draw FX perpendicular to the directrix. Then FX is the axis of the parabola. To complete the curve, take any length FP_2 as radius, and draw the arcs P_1, P_2. Set off along the axis from X, $\text{XN}_2 = \text{FP}_2$, and through N draw a line perpendicular to the axis, cutting the arcs in P_1 and P_2. These are points on the curve, for $\text{FP}_2 = \text{XN}_2 = \text{P}_2\text{N}_1$. Repeat the construction for as many points as required, and join up the points, making a continuous curve, which is the parabola.

---

* Consult Wilson's "Solid Geometry and Conic Sections," p. 93, or Eagle's "Plain Curves."


G
F
A
C
D
M


P₁
P
M
F
X
N₂
N
Q
N₁
P₂

No. 52, 1893. **F is the focus of a parabola, AB is a tangent to the curve, and A is on the directrix. Find the axis and directrix, and draw a sufficient portion of the curve to show that AB is truly tangent to it.**

The line AB and F the focus (Fig. 1) are given. Commence by joining A to F, and draw FP perpendicular to FA, meeting the tangent in P; P is the contact point of tangent with the curve. The directrix may readily be found, for it will be a tangent to the circle KD*g* struck with centre P and radius PF. The axis may now be drawn through F and perpendicular to the directrix. Points in the curve may be obtained as shown in previous figure.

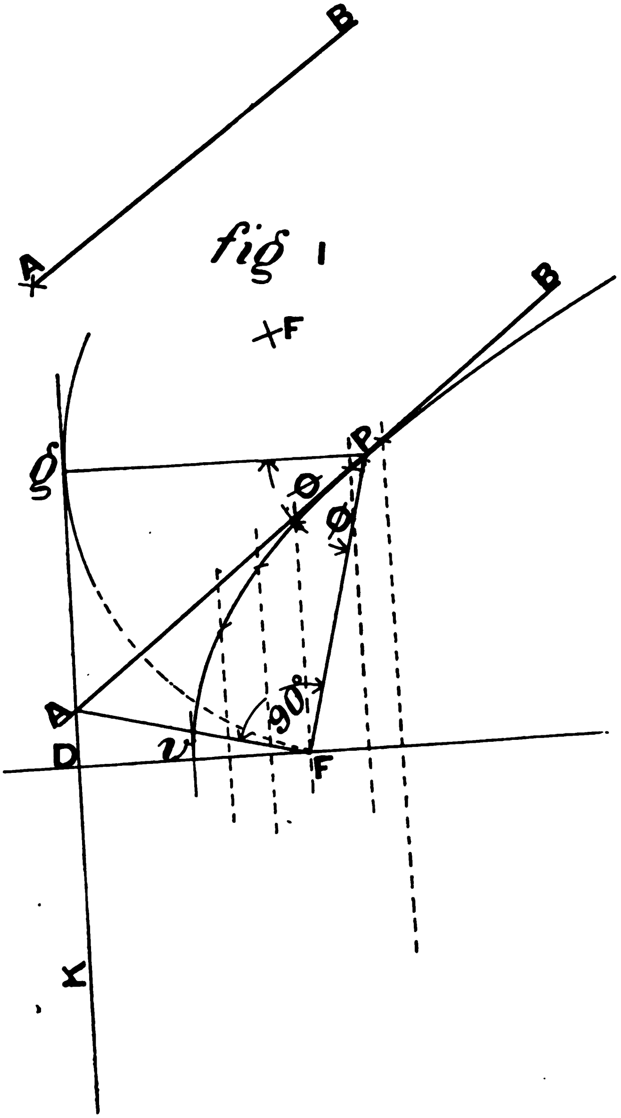
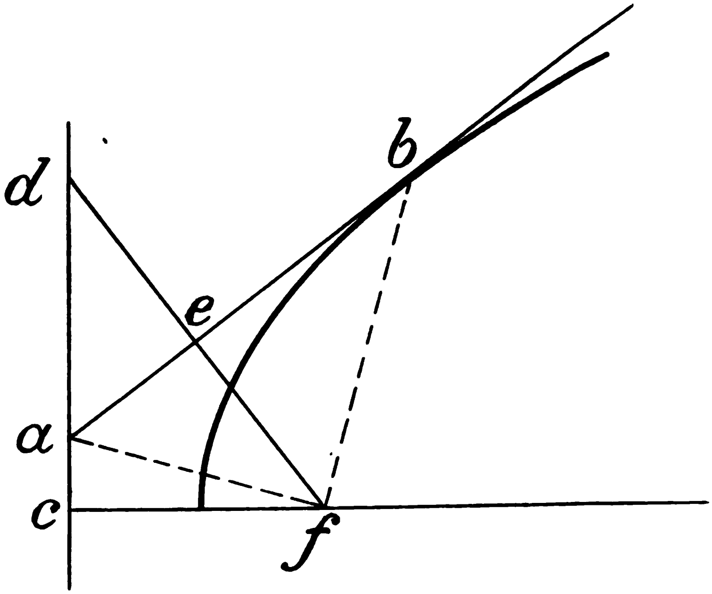


fig 1

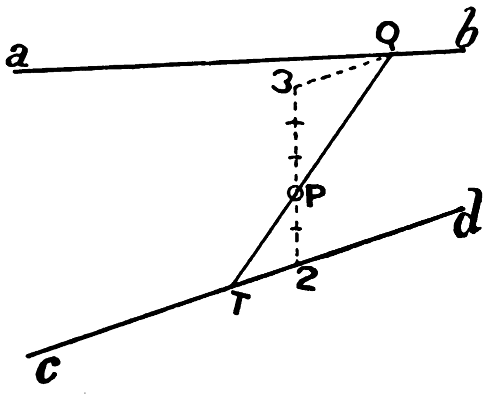


Second method.—

From the focus *f* drop a perpendicular on to the tangent *ab*, cutting it in *e*. Produce *fe* so that *ed* = *fe*, then *d* is a point on the directrix.* Join *da*, which is the directrix. From *f* drop a perpendicular on to the directrix, cutting it in *c*. Then *cf* is the axis. With the axis, directrix, and focus the curve can be described. The contact point is *b* where *fb* cuts the tangent; *fb* is at right angles to *af*.

---

* Consult Wilson's "Solid Geometry and Conic Sections," p. 98.



Through the given point P draw a line terminated by the given lines *ab*, *cd*, and divided at P into two parts which are as 2 : 3.

Draw through P any line such as 2P3; let P2 = 2 parts, and cut off P3 = 3 of these parts; then through 3 draw 3Q parallel to *cd*. Then draw the line QPT, which is divided at P into two parts as 2 : 3.

For the △ PT2 is similar to the △ PQ3, hence P2 : PT : : P3 : PQ: (Euc. VI. 2.)

E

o. 52, 1891. **Construct a triangle ABC; AB = 2 in., BC = 1·5 in., CA = 1·2 in. Produce AB to P, 1 in. from B. Draw a parabola touching the three sides of the triangle, and passing through P.**

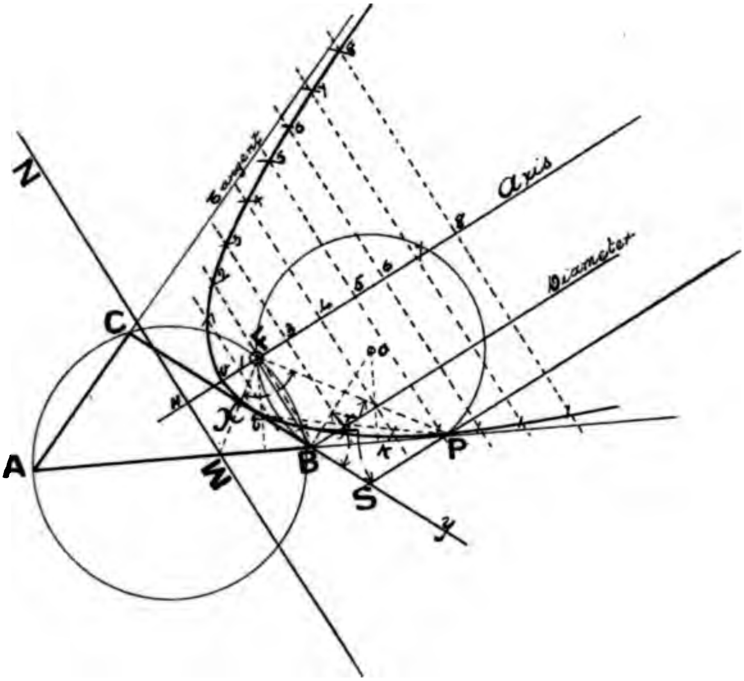
N.B.—The external angle between any two tangents to a parabola is equal to the angle which either subtends at the focus.

Describe a circle passing through ABC; the focus F will lie on this circle.* On BP describe a segment of a circle to contain an angle equal to the exterior angle between the two tangents meeting at B, *i.e.* the ∠ PBy; this segment will cut the first circle at point F, which is the required focus. This segment may be found by drawing BO at right angles to By, meeting the perpendicular bisector of BP in O. Then *o* will be the centre. The direction of axis may be found, since it is parallel to the diameter, which is found by joining F to P, and making ∠ BPS = ∠ BPF, or make ∠ BFx = ∠ BFP; x will be point of contact of tangent CB with the curve. Join B to r, the centre point of tP; then Br will be a diameter. The directrix may be found by drawing FM perpendicular to FP, meeting AB in M; then through M draw directrix perpendicular to axis. The vertex may be found, since it is the centre point between the focus and the directrix.

Points on the curve may be found by drawing a series of ordinates such as 1 1′ . . . 8 8′, and taking in your compasses the distances from the directrix in each case, such as 1H, and with F as centre describe an arc cutting ordinate 1 1′; with H2 as radius and F as centre, draw an arc cutting ordinate 2 2′; and so on. All these points will be points on the curve, because the curve of a parabola is such that the distance of any point on the curve to the focus is equal to the distance of that point to the directrix.

---

* Eagle's "Plain Curves," Prob. 50.

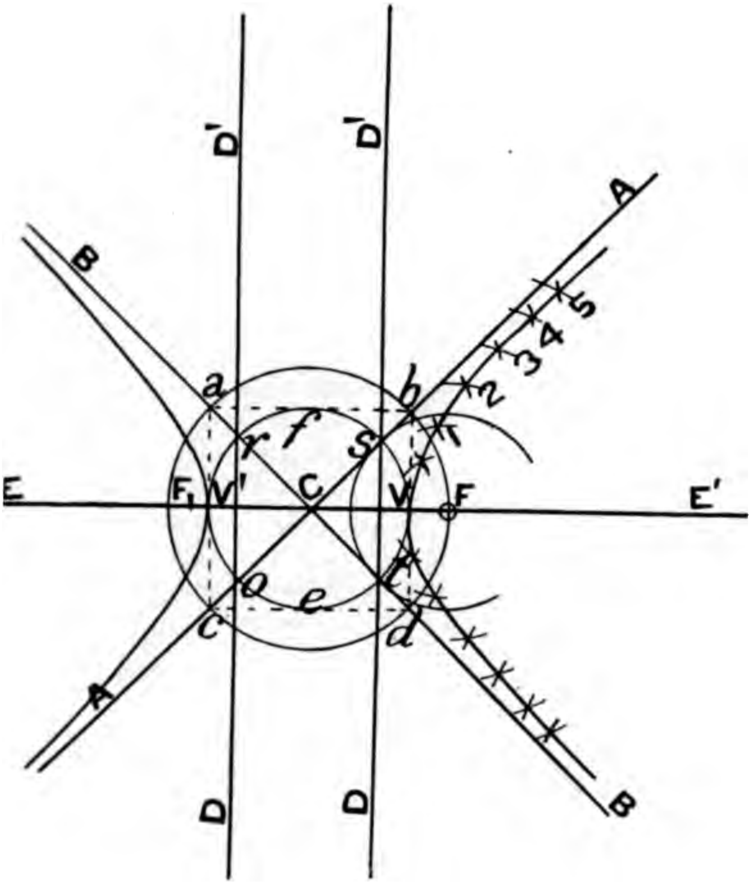

N
C
A
M
B
S
y
P
F
X
t
H
W
k
oo
Tangent
Axis
Diameter
1
2
3
4
5
6
7
8

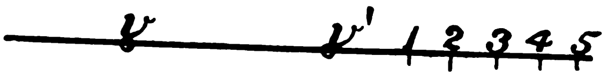
No. 42, 1887. **Draw a rectangular hyperbola, having given the focus F and the asymptote AA.**

The angle between the asymptotes of a rectangular hyperbola is a right angle,* *and all its axes are equal.* With F, the given focus, as centre, describe a circle tangential to the given asymptote AA. Draw BB, the other asymptote, perpendicular to AA and tangential to the circle; then the axis EE′ will bisect the ∠ ACB. With C the crossing-point of AA and BB, and radius CF describe a circle, cutting EE on the left in point F_1 (F_1 is the other focus), and cutting AA and BB in points *a*, *b*, *c*, and *d*. Join *ab* and *cd*. Then *ab* and *cd* are tangents to the curves at their vertices, and V′, V are the vertices of the curves on the left and right respectively; the directrices D′DD′D will pass through the crossing-points *o*, *r*, *s*, *t* of the circle struck with C as centre and radius CV. The curve may readily be described thus: Draw a line *vv′* . . . 5, and make *vv′* equal to the transverse axis V′V. Take any length *v*1 as radius, the focus F_1 as centre, and describe the arc 1. With F, the given focus, as centre, and radius 1*v′*, describe an arc cutting the first one in the point 1, and the same with radius *v*2. 2*v′*, centre F_1F, find point 2; points 1 and 2 are on the hyperbolic curve. For $F_1 2 - F2 = V'V$, the transverse axis, and $F_1 1 - F1 = V'V$. A series of points found similarly, and the curve drawn through them will be the required curve.

---

* See Eagle's "Plain Curves," p. 190.


D'
D'
A
B
a
b
r
f
s
1
2
3
4
5
E
F₁
V'
C
V
F
E'
o
e
t
c
d
A
B
D
D


v
v'

52,
892.

Determine the locus of a point which moves so that the lines joining it to two given points AB make equal angles with the given straight line CD. What is this locus?

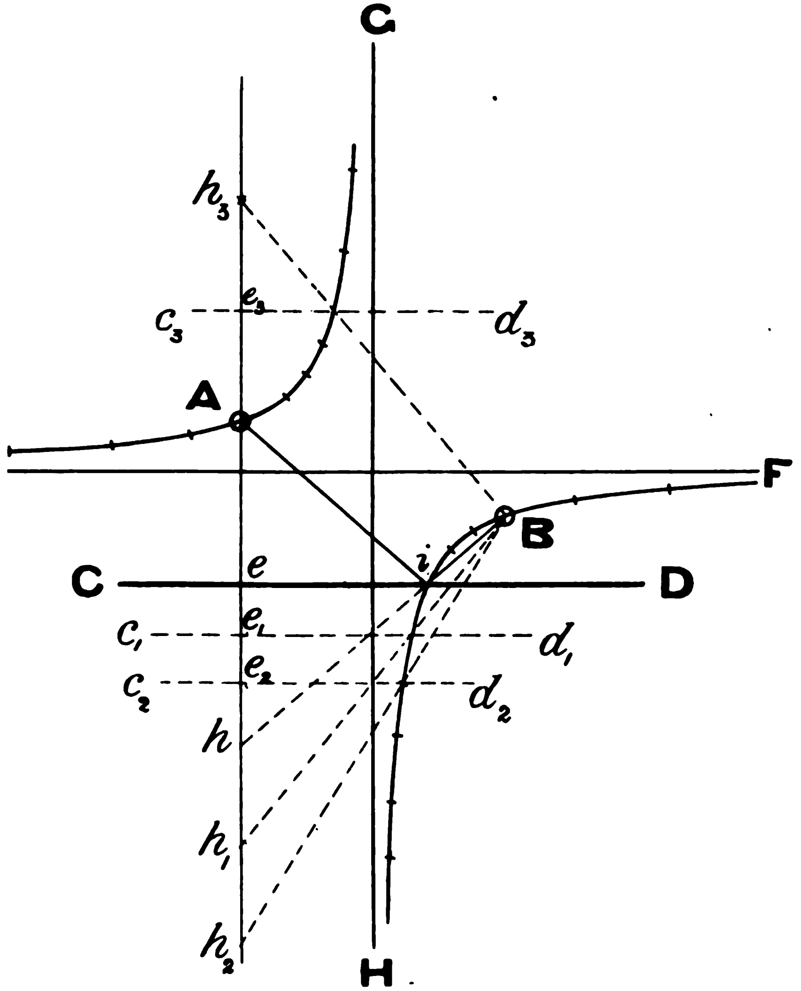
The virtual image of A in CD is at h. Join hB, cutting CD in i, and draw Ai. Then Ai and Bi make equal angles with CD.* Draw any other line c_1d_1 parallel to CD. The virtual image of A in c_1d_1 is at h_1, as far below e_1 as A is above it. Join h_1B; then the point of intersection of c_1d_1 and h_1B is a point on the locus, because c_1d_1 is parallel to CD. Similarly, take a third line c_2d_2 parallel to CD. The virtual image of A in CD is at h_2, and the point of intersection of h_2B with c_2d_2 is a point on the locus. In the same way the point of intersection of h_3B with c_3d_3 is a point on the locus, when h_3 is the virtual image of A in c_3d_3.

It will be noticed that the locus continually approaches the lines GH and EF. These lines are asymptotes to the curve, which is a rectangular hyperbola.

It may be proved thus: If it can be shown that, in a rectangular hyperbola in which A and B are the extremities of a diameter, the inclinations of Ai and Bi to CD are equal, then the curve found in the problem must be a rectangular hyperbola. Join AB, cutting EF and GH in X, and draw a tangent to the curve at B, cutting Ai in N, and EF in S; then it is a property of a rectangular hyperbola (see Besant's "Conic Sections," p. 135) that the angle NBi between the tangent NB and the chord Bi equals the angle BAi between the diameter BA and the line Ai. Now, the inclination of Bi to EF (*i.e.* to CD) = NBi + ESB, and the inclination of Ai to EF = AXE + XAN; and as ESB = BXS = AXE, the above inclinations are equal, and therefore the curve found above is a rectangular hyperbola.

---

* For $eih = ei$A, and $eih =$ BiD (Euc. I. 15); ∴ BiD = Aie.


G
h_3
e_3
c_3
d_3
A
F
B
i
e
C
D
e_1
c_1
d_1
e_2
c_2
d_2
h
h_1
h_2
H

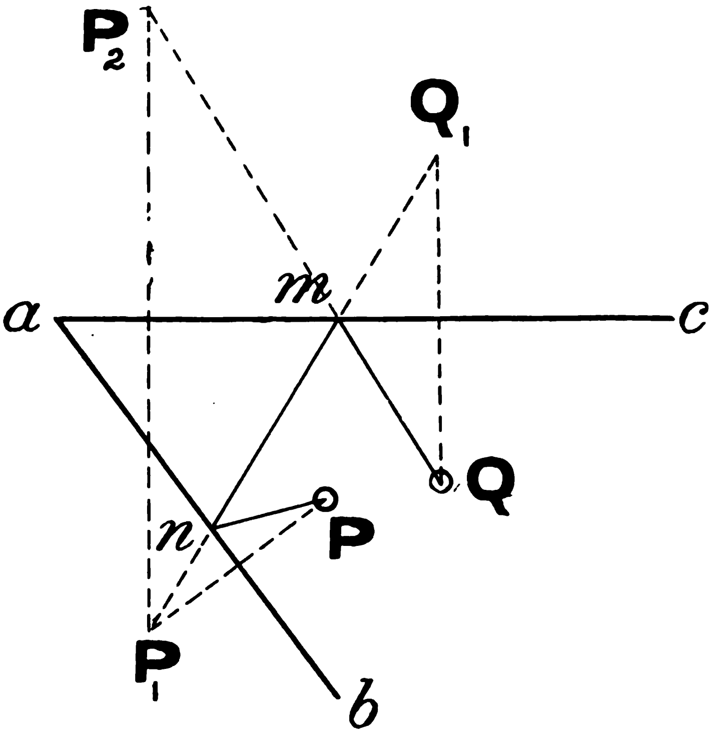
No. 53, 1892. **A ray of light from P is reflected to Q after impinging successively on *ab* and *ac*. Assuming the angle of incidence to be equal to the angle of reflection, trace the path of the ray.**

The virtual image of P in *ab* is at P_1, as far on the other side of *ab* as P is in front of it;* and hence the reflection of P will appear (to any one looking into the mirror) as if it were at P_1. Similarly, the virtual image of P_1 in *ac* is at P_2, and an eye placed at Q must look along the line P_2Q to see it. Hence join *m* (where P_2Q cuts *ac*) to P_1, cutting *ab* in *n*, and join *n*P. The path of the ray is P*nm*Q.

Or thus: as the angle of incidence equals the angle of reflection, Q_1mc equals Qmc where Q_1 is the virtual image of Q in *ac*. An eye placed at Q_1, looking along Q_1m, will see exactly the same as an eye placed at Q and looking along Qm. But an eye at Q_1 sees P_1 along the line Q_1P_1, cutting *ac* in *m* and *ab* in *n*. Hence P*nm*Q is a path of the ray of light.

---

* *Vide* any treatise on optics or light.


P2
Q1
m
a
c
Q
n
P
P1
b

No. 51, 1894. **Draw a rectangle, sides 3 in. and 1¾ in. Suppose this rectangle to represent a garden. It is required to trace round the garden a path of *uniform width*, leaving in the centre a rectangular plot one-half the area of the garden. Draw this plot.**

Let the rectangle ABCD be of the required size, and produce BC to d'' so that $Bd' = BD$, and $Bd'' = BC + CD$.

Then divide $d'd''$ into four equal parts; then one of such parts will be the width of the required path round the garden.

For let breadth of path $= x$.

$$\text{Then length of inside rectangle} = a - 2x$$

Likewise breadth of inside rectangle $= b - 2x$ (calling side BC a and side CD b).

Then—

$$(a - 2x)(b - 2x) = \frac{ab}{2} \text{ by hypothesis}$$

which simplifies to—

$$4x^2 - 2x(a + b) = -\frac{ab}{2}$$

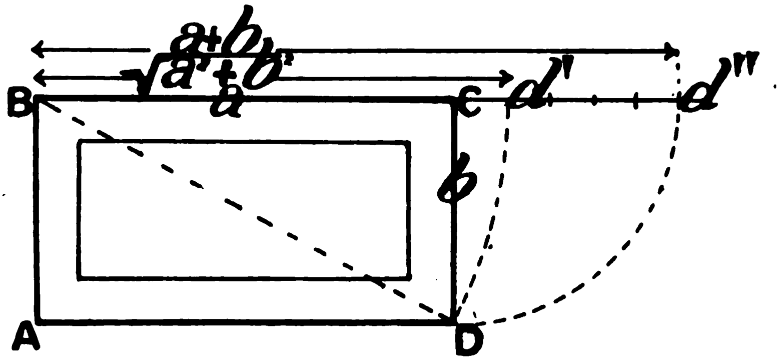
Dividing by 4 and adding $\left(\frac{a + b}{4}\right)^2$ to both sides, we get—

$$x^2 - \frac{x(a + b)}{2} + \left(\frac{a + b}{4}\right)^2 = \frac{(a + b)^2}{16} - \frac{2ab}{16} = \frac{a^2 + b^2}{16}$$

Taking square root of both sides, we get—

$$x - \frac{a + b}{4} = \frac{\pm\sqrt{a^2 + b^2}}{4}$$

$$x = \frac{a + b \pm \sqrt{a^2 + b^2}}{4}$$



When the upper sign is taken before the radical, the value of x so derived is the width of a path described round the garden outside the rectangle ABCD, such that ABCD is half the area of the whole. When the lower sign is taken the path is described inside the rectangle ABCD, as shown in the figure, such that the area of the smaller rectangle is half that of ABCD; for BD $= \sqrt{a^2+b^2} =$ Bd', and Bd'' = BC + CD. And—

$$x = \frac{\text{B}d'' - \text{B}d'}{4}$$

the width of path required.

CHAPTER II.

GRAPHIC STATICS.

No. 65, 1892. **In the differential axle (see sketch), assuming that the diameters A, B, and C are 2 ft., 1 ft. 1 in., and $8\frac{1}{2}$ in. respectively, and that there is no friction, determine and write down the weight which must be applied at P to balance 150 lbs. at W.**

An end view of the contrivance is shown in the figure. The load W is supported by the two ropes attached to the lower block, hence the tension in each will be $\frac{W}{2.}$ By the second law of equilibrium, the sum of the moments of all the forces acting on the top block about the centre of the bearing must be zero; or—

$$PR + \frac{W}{2} r_2 - \frac{W}{2} . r_1 = 0$$

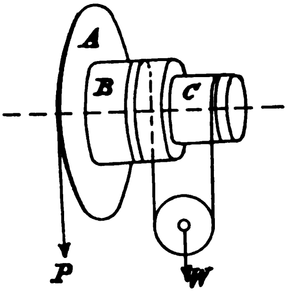
$$\text{and } P = W . \frac{r_1 - r_2}{2R}$$

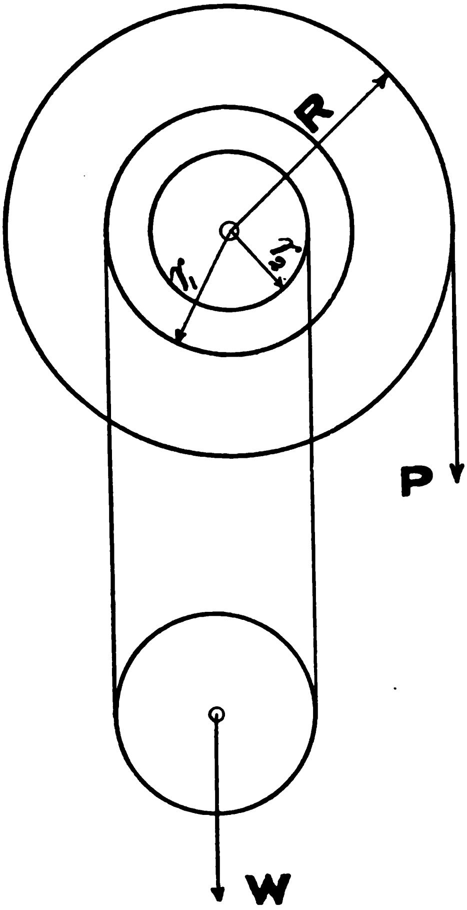
Set down *ma* equal to $r_1 - r_2$, and *mb* equal to 2R; also *md* at any angle to represent W. Join *bd*, and through *a* draw *an* parallel to *bd*. Then *mn* = P; for by Euc. VI. 2—

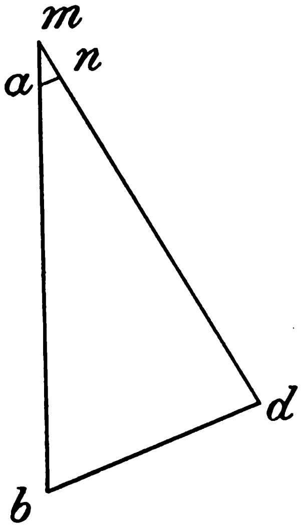
$$\frac{ma}{mb} = \frac{mn}{md}, \text{or } \frac{r_1 - r_2}{2R} = \frac{P}{W}$$

mn scales off as 14 lbs.

For a solution of this problem by means of the funicular polygon, see the Miscellaneous Examples.


A
B
C
P
W

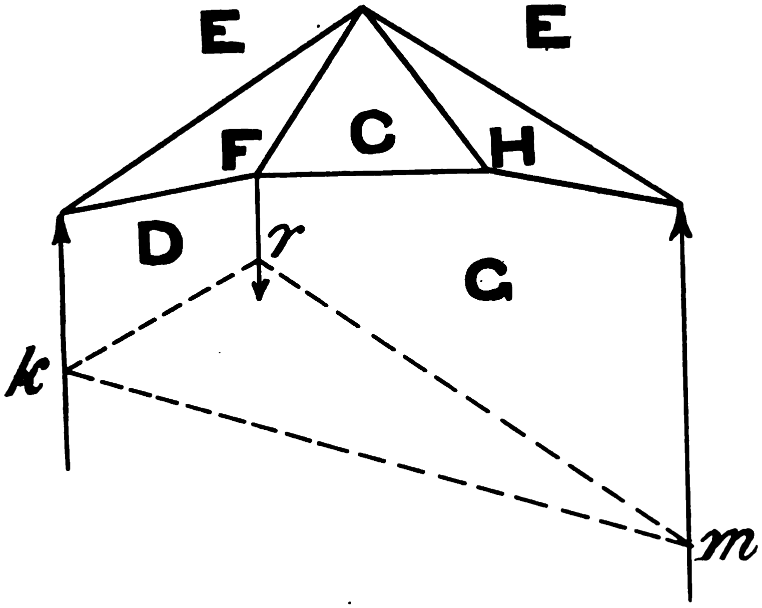

R
r_1
r_2
P
W

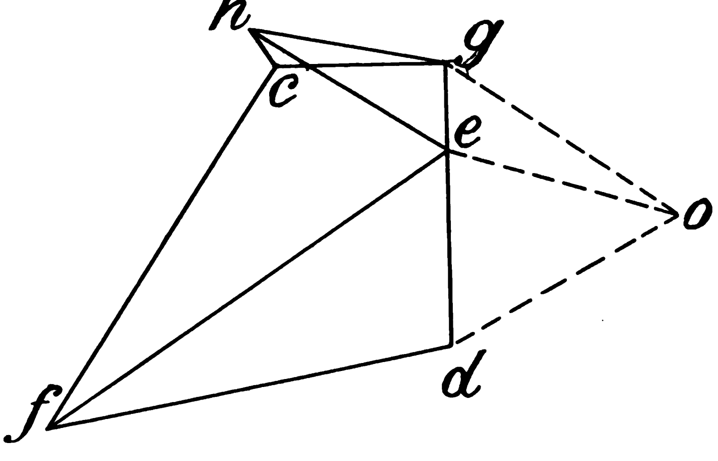

m
n
a
d
b

No. 55, 1887. **Draw a stress diagram for the given simple truss under a single load of 1800 lbs. applied as shown. Distinguish between bars in tension and in pressure (scale $\frac{1}{8}$ in. to 100 lbs.).**

Label all the spaces between the external forces, such as D, E, G. Set down the line of loads *gd* to the proper scale, and label the spaces between the members of the structure. To find the reactions DE and EG, take any point O as pole, and draw the dotted lines *og* and *od*. Through the space D and parallel to *od* draw *kr* intersecting DE and DG in R and *t*; then through *r* draw *rm* through the space G, a line parallel to the line *og*. Join *km*, and through *o* draw *oe* parallel to *km*, cutting *gd* in *e*. Then *de* is the reaction on the left, and *eg* that on the right. Now draw the stress diagram. Through *d* draw *df* parallel to DF, and *ef* parallel to EF, intersecting in *f*: then *ef* and *df* represent the stresses in EF and DF respectively. Further, through *f* draw *fc* parallel to FC, and through *g* draw *gc*, parallel to GC, and so on until every member is represented in the stress diagram.

To find the kind of stress in each member, take a joint such as the point of application of DG. That force acts downwards, and therefore is represented in direction by *gd*. The polygon of forces determining the equilibrium of the joint is *gdfcg*. If we place arrow-heads on each side of the polygon they will all have the same sense, or follow each other round the polygon. These directions are the directions in which the several members act upon the pin of the joint. Following round the polygon, the arrow-head (not shown) on *df* points from *d* to *f*, therefore the member DF acts on the pin of the joint in that direction, that is it pulls it, consequently DF is in tension. Similarly, FC is in tension; the same reasoning applies to every member. The positive sign indicates tension, because the member will increase in length under that tension. The negative sign indicates compressive stress. The stresses in pounds are—


E
E
C
F
H
D
r
G
k
m


h
g
c
e
o
d
f

Member.	Stress.	Member.	Stress.	Member.	Stress.
fd	+ 2600	*de*	1200	*gc*	+ 1075
fe	− 3100	*eg*	600	*gh*	+ 1250
cf	+ 2700	*eh*	− 1450	*hc*	+ 300

No. 65, 1894. **A roof-truss of given form is loaded on one side only with a total uniformly distributed normal pressure of 2½ tons due to wind. The foot at Q is fixed; the other at N supported only. Determine and write down the stresses in the respective members of the truss, distinguishing tension and pressure.**

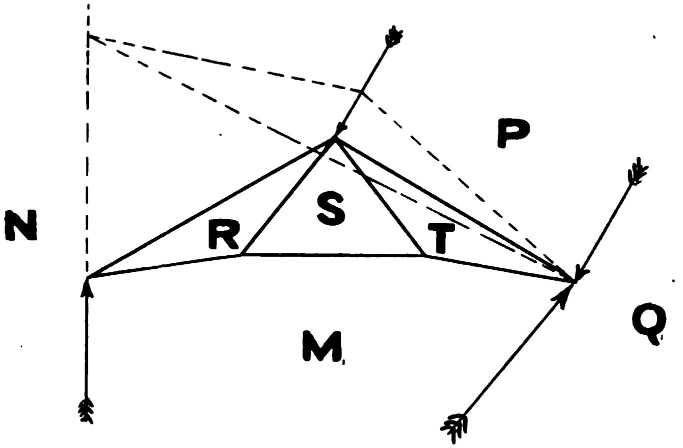
Label all the spaces between the forces as shown in the figure, and draw a line of loads as far as possible. Such is the line *qp*, *pn*, and a vertical through *n*. Take any pole *o*, and draw the radial lines *oq*, *op*, *on*, and then draw the funicular polygon shown dotted, beginning at the point of application of MQ, viz. the right-hand end of the truss.* Draw the closing line of the funicular polygon, and through *a* draw a line parallel to it, cutting the vertical through *n* in *m*. Join *mq*; then *mq* is the magnitude and direction of the force MQ.

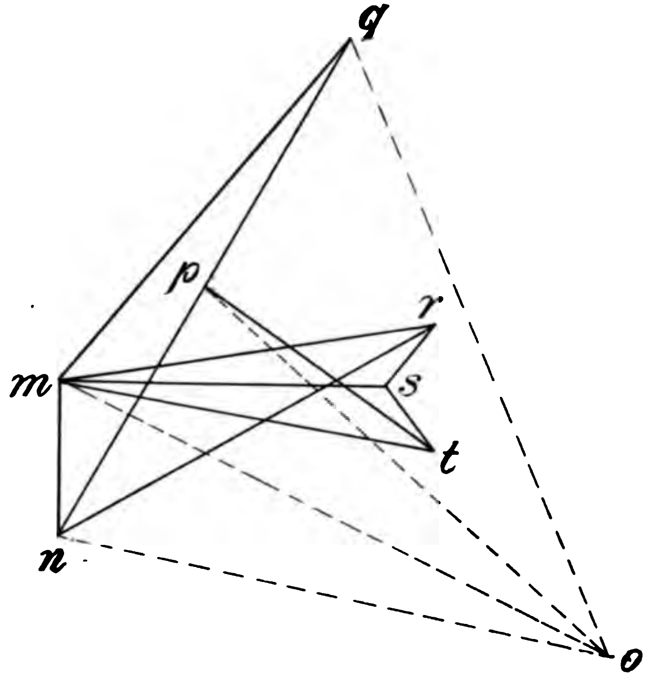
The stress diagram can now be drawn in the usual way. The following are the stresses, + denoting tension, and − compression:—

Member.	Stress in tons.
NM	0·7
MQ	1·95
NR	− 1·7
MR	+ 1·65
PT	− 1·25
SM	+ 1·42
SR	+ 0·35
ST	+ 0·35
MT	+ 1·65

---

* See "Graphic Methods of Structural Design," by W. W. F. Pullen, p. 38.


N
R
S
T
P
M
Q


q
p
r
m
s
t
n
o

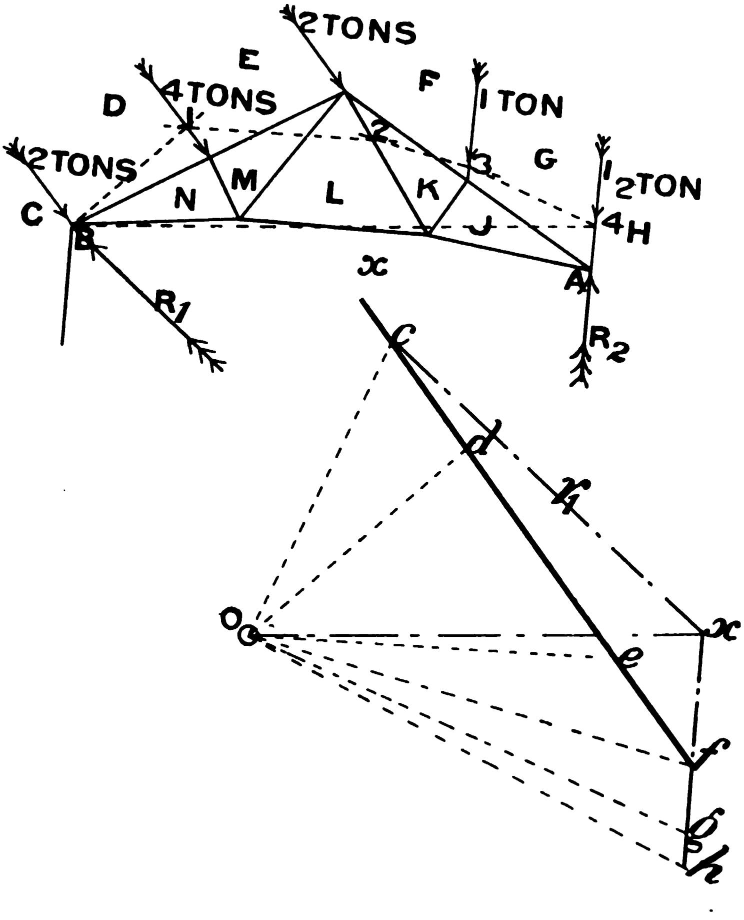
No. 55, 1889. **The given roof-truss is loaded as shown. The reaction at A is vertical. Determine and write down its magnitude, and the direction and magnitude of the reaction at B.**

Commence by labelling each space between the external forces, as CDEFGH, so that each force may be known by the letters on each side of it; thus the force on the vertex of the roof is called EF. Next draw the load-line, making *cd* = 2 tons and parallel to force CD, *de* = 4 tons, *ef* = 2 tons, *fg* = 1 ton, *gh* = ½ ton, to any convenient scale and parallel to their respective forces. Select any pole *o*, and draw vectors *co*, *do*, . . . *ho*. Now draw the funicular polygon B1234 about the forces parallel to these vectors in turn, thus B1 parallel to *do*, meeting force DE in 1, 1 2 is parallel to *do*, meeting force EF in 2, etc., the last link meeting force GH in point 4. Join 4 B; this line is the closing line of the polygon. From *o*, the pole of the polar polygon, draw *ox* parallel to 4B, the closing line, meeting the vertical *hf* in *x*; then *hx*, measured to the same scale, is the reaction at A. Join *x* to *c*; then *xc* to the same scale gives the magnitude and direction of the reaction at B.*

NOTE.—Should the stresses in each member be required, letter each space of the truss as shown, and from each corresponding point in the polar polygon draw the corresponding lines parallel to the members of the truss; thus, from *d* draw *dn* parallel to DN, and through *x* draw *xn* parallel to XN. These lines intersect in *n*; then *dn*, *xn* represent the stresses in those members, and so on for all the members.

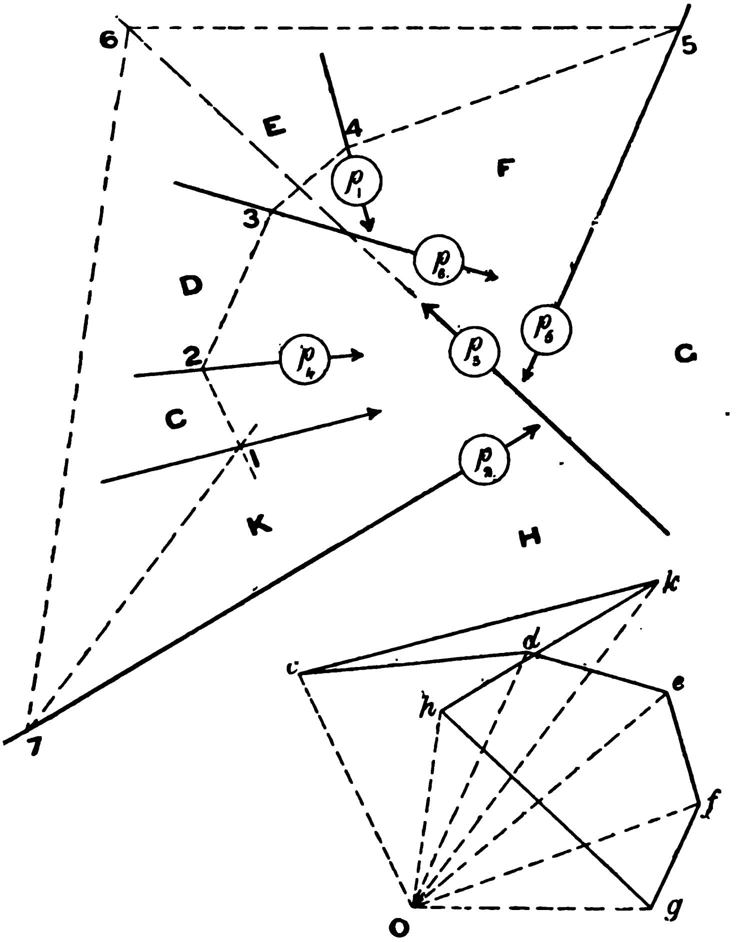
---

* See Pullen's "Graphic Methods of Structural Design," p. 36.


2 TONS
E
F
1 TON
D
4 TONS
1
2
3
G
½ TON
2 TONS
M
L
K
N
C
B
J
4
H
x
A
R1
R2
c
d
r1
O
x
e
f
g
h

No. 64, 1894. **Determine, by means of a polygon of forces and a funicular polygon, the magnitude and direction of the resultant of the six given forces, $p1$ . . . $p6$ ($p1$ = 165 lbs., $p2$ = 355 lbs., $p3$ = 410 lbs., $p4$ = 323 lbs., $p5$ = 162 lbs., $p6$ = 205 lbs.), using a scale of ½ in. = 100 lbs.**

The lines of action of the forces p_1, p_2, etc., are given, and it will be found less confusing to arrange them (by producing them in either direction) so that the arrow-heads are near the centre of the figure. Label the spaces between the forces as in the figure, viz. C, D, E, F, G, H, K. The first and last letters, apparently, are in the same space. Beginning with c, draw the line of loads or force polygon $cdefghk$; then ck = 520 lbs. is the resultant of all the forces p_1, p_2, p_3, etc. To obtain the position of the line of action of the resultant, take any pole O and draw radial lines, Oc, Od, etc. Then anywhere across the space C draw a line 1 2, parallel to Oc, cutting p_4 in 2. Through 2 and across the space D draw a line parallel to Od, cutting p_6 in 3, and repeat this construction until 7 1 is drawn across the space k parallel to Ok. The lines 7 1 and 2 1 intersect in 1. Then the line of action of the resultant passes through 1. Hence through 1 draw a line parallel to ck; this is the position of the line of action of the resultant.


6
5
E
4
F
p_1
3
p_6
D
p_8
p_5
G
2
p_4
C
1
p_2
K
H
7
k
d
c
e
h
f
g
O

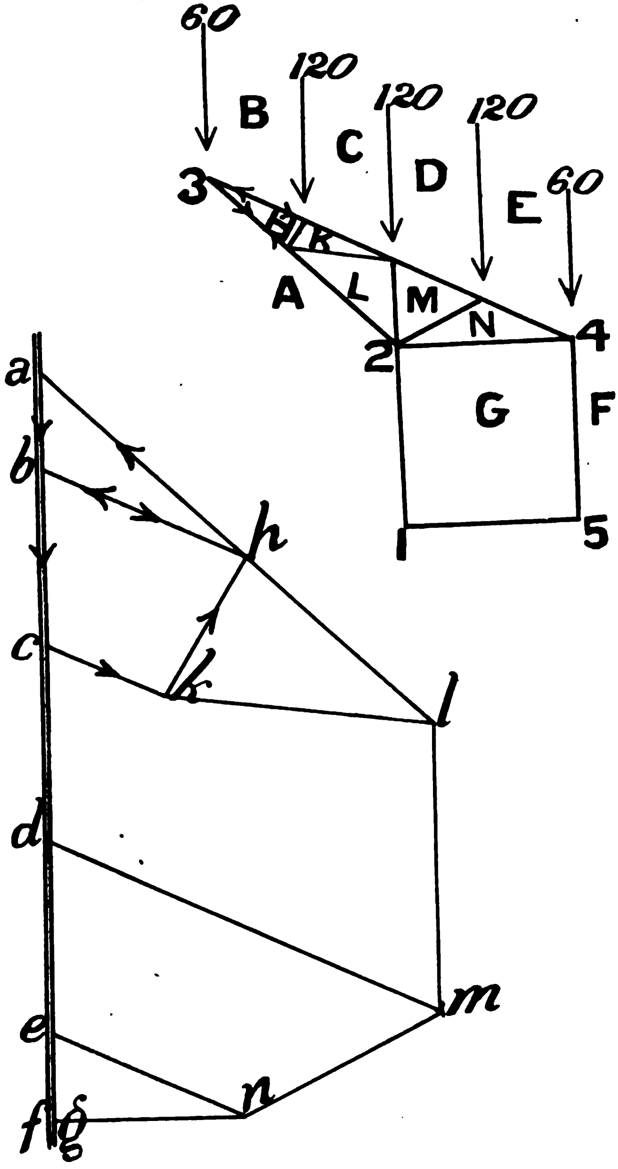
No. 65. 1893. **1 2 3 4 5 is the section of an open shed. The projecting roof 2 3 4 rests on supports 1 2, 4 5 fixed at 1 and 5; 3 4 is equally divided, and HK is perpendicular to 3 4. The loads are vertical and represented in pounds. Find and write down the stresses on the members of which the roof is composed; and indicate the nature of the stresses, whether tensile or compressive. In constructing the polygon of stresses, employ by preference for unit 1 in. to 100 lbs.**

Commence by labelling each space as shown.

Next draw the load-line *a* . . . *f* equal to the total load, making *ab* equal to the load which is between AB, and so on for each of the loads.

Then through *a* draw *ah* parallel to AH, and *bh* parallel to BH, meeting in point *h*: then *bh*, *ah* represent to the same scale the stresses in the members BH and AH respectively. Similarly, for every member of the truss, we draw *hk* parallel to HK, and *ck* parallel to CK, meeting in *k*; draw *kl*, *al* parallel to KL and AL, meeting in *l*; draw *bm*, *am* parallel to LM and DM, meeting in *m*; draw *mn*, *en* parallel to MN and EN, meeting in *n*; then *fg*, *ng* parallel to FG, NG, meeting in *g*. *f* and *g* being coincident, there is no stress in GF. We get the amount of stress in each member, by measuring lengths in the stress diagram.

The nature of the stress will be readily found if we consider one joint at a time; and remembering that the arrow-head must run in the same direction round the polygon of forces for that joint. Let us consider the joint 3, where a load of 60 lbs. is applied. Its arrow pointing downwards. The load is shown on the stress diagram at *ab*, and the members AH, BH at *ah* and *bh*. Place the arrow-head on *ab*, pointing downwards towards *b*; on *bh*, pointing towards *h*; on *ha*, pointing towards *a*. Place these arrows now about the joint 3 in the same order, when it will be seen that HA is in compression, and BH in tension. Now proceed to the next joint and continue the same reasoning for the load on it; thus the arrow representing the load *bc*


60
120
120
120
60
B
C
D
E
3
H
K
A
L
M
N
2
4
G
F
1
5
a
b
h
c
k
l
d
m
e
n
f
g

points to *c*, then *c* to *k*, *k* to *h*, *h* to *b*, and similarly for every joint. The appended table shows the amount and nature of the stresses in the given truss:—

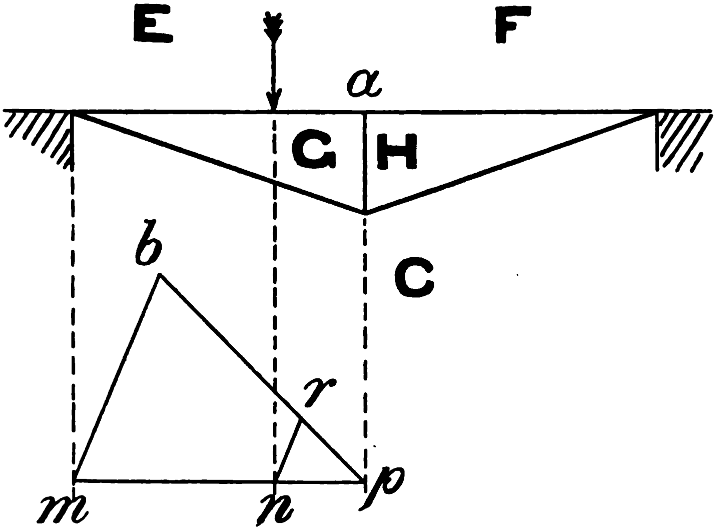
Member.	Stress in lbs.	Nature.	Member.	Stress in lbs.	Nature.
AH	170	Compression	MD	260	Tension
BH	135	Tension	MN	130	Compression
HK	100	Compression	NE	140	Tension
KC	80	Tension	NG	130	Compression
KL	170	Tension	GF	0	0
LA	330	Tension	GA	480	Compression
LM	180	Compression			

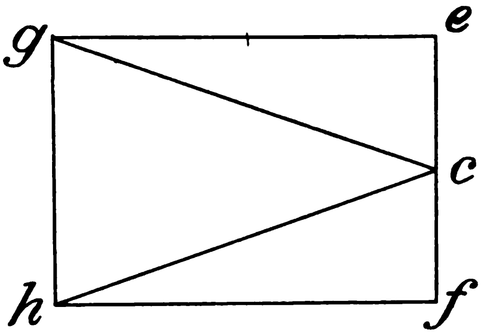
No. 64, 1892. **The figure represents a trussed beam of 20 ft. span. There is a uniformly distributed load of 300 lbs. per foot run, and a concentrated load of 1000 lbs. applied at point shown by the arrow. Determine and write down the stresses in the members of the beam, distinguishing between tension and pressure.**

The load of 1000 lbs. is supported by the member EG, a hinge being assumed at the point *a*. First find how much of the 1000 lbs. is supported at each end of EG. Draw the horizontal line *mp* parallel and equal to the length of the member EG. Divide it in *n* as shown. Set up at any angle the line *pb*, representing 1000 lbs.; join *bm*, and through *n* draw *nr* parallel to *bm*. The part *br* represents 700 lbs., the portion of the 1000 lbs. supported at *a*. The distributed load is 6000 lbs., half of which will be concentrated at *a*. We then have at *a* 3000 + 700 = 3700 lbs. Set down the line of loads *ef* to represent 3700 lbs., and bisect it in *c*. Complete the stress diagram and scale off the stresses. They are—

Member.	Stress in lbs.	Member.	Stress in lbs.
GH	− 3700	EG	− 5300
HC	+ 5600	FH	− 5300
GC	+ 5600		

A trussed beam is very different from the assumptions made above. The question is in reality very misleading, and the trussed beam can only be solved by the method of work (see "Graphic Methods of Structural Design," by W. W. F. Pullen, p. 296).


E
F
a
G
H
C
b
r
m
n
p


g
e
c
h
f

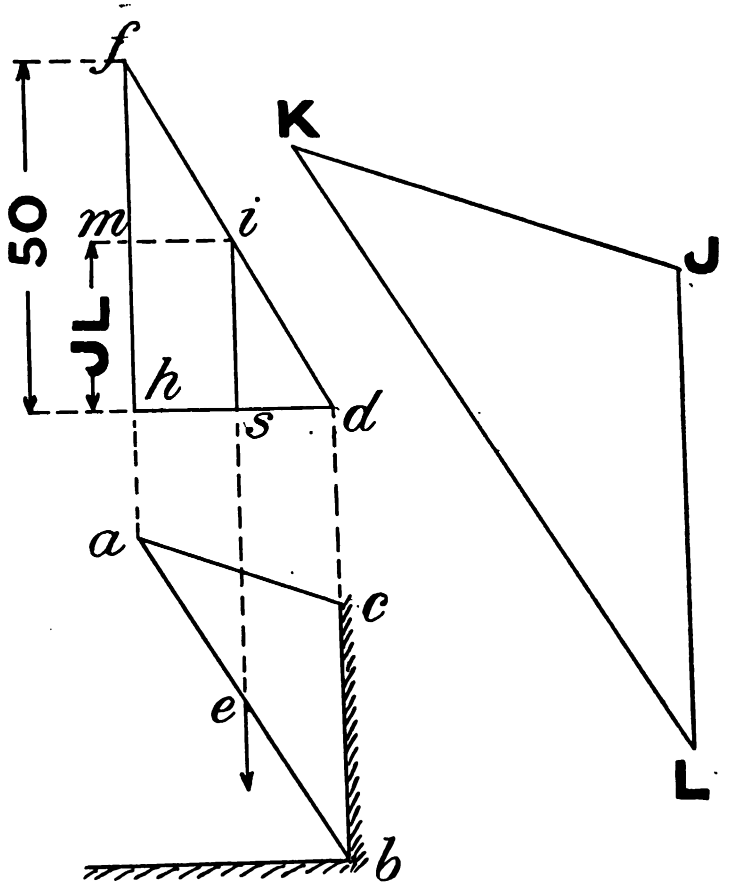
No. 55, 1888. **The line *ab* represents a rigid rod, and *ac* a tie. At what point along *ab* must a load of 50 lbs. be attached so that the tension in *ac* is 29 lbs.?**

First find what load applied at *a* will produce a tension of 29 lbs. in *ac*. This load is JL when KJ represents 29 lbs. Now find the point in *ab* at which 50 lbs. will have the same moment about *b* that JK applied at *a* will have about *b*. In other words, find *sd* in the equation—

$$\mathrm{JL} \times hd = 50 \times sd$$

which can be written $\frac{hd}{sd} = \frac{50}{\mathrm{JK}}$.

Through *a* and *c* erect verticals. Take any point *h* in *ah*, and draw *hd* horizontal. Set up $hf = 50$ lbs. and $hm = \mathrm{JL}$ lbs. Through *m* draw *mi* parallel to *hd*, and through *i* draw *isc* parallel to *fa*. Then, from similar triangles, $\frac{fh}{is} = \frac{hd}{sd}$, the same as required above; consequently the point *c* is the position of the 50 lbs.


f
K
50
m
i
J
JL
h
d
s
a
c
e
L
b

No. 61, 1895 **The diagram represents a swivel bridge, its pivot placed at NL. The loads, *in tons*, are placed, as indicated below, at CQ, QP and PN, above at CD, DE and EF. Determine the equal loads on the counterpoise to the right of NL, placed above at FG and GH, below at HK and KL.**

For the construction of the funicular polygon, take 1 ton = 0·4 in.

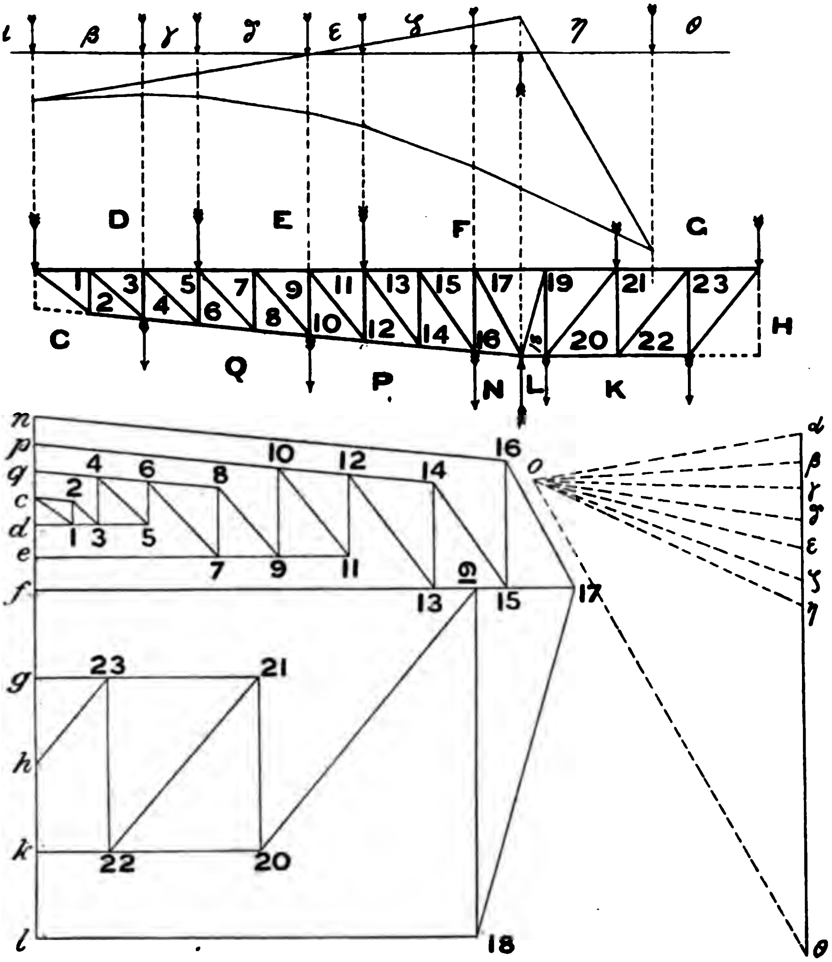
Find and write down the stresses in the members of the bridge.

The original truss is shown in thick lines in the figure, supported by the single force NL (using Bow's notation) and loaded with the following loads: NP = PQ = QC = CD = 1 ton, DE = EF = 1¼ ton, and the four equal but unknown loads, FG, GH, HK, KL. The members shown dotted are useless and sustain no stress. The value of each of the unknown loads must first be found. Their centre of gravity, or, more properly, centre of force, will be at the middle of the second panel from the end. A horizontal line has been drawn at the top of the figure, and all the loads are shown by arrows above this line, and the single supporting force below the line. Next put down the line of loads so far as they are known, viz. $\alpha\beta\gamma\delta\epsilon\zeta\eta$, to the given scale of 2·5 tons to the inch. Next draw the funicular polygon in the upper part of the figure, in which the lines of the polygon are parallel to the radial lines $O\alpha$, $O\beta$, etc. Draw the closing line of the polygon across the space θ, and through O in the pole diagram draw $O\theta$ parallel to the above closing line. Then $\eta\theta$ is the resultant of the four unknown loads. Set down a new line of loads, *nl*, to avoid confusion, and draw the stress diagram in the usual manner.*

The following table contains the stresses in the individual members of the truss, found from the stress diagram:—

---

* *Vide* "The Application of Graphic Methods to the Design of Structures," by W. W. F. Pullen, p. 38 *et seq.*


ι
β
γ
δ
ε
ζ
η
θ
D
E
F
G
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
H
C
Q
P
N
L
K
n
p
q
c
d
e
f
g
h
k
l
o
16
17
18
19
20
21
22
23
ι
β
γ
δ
ε
ζ
η
θ

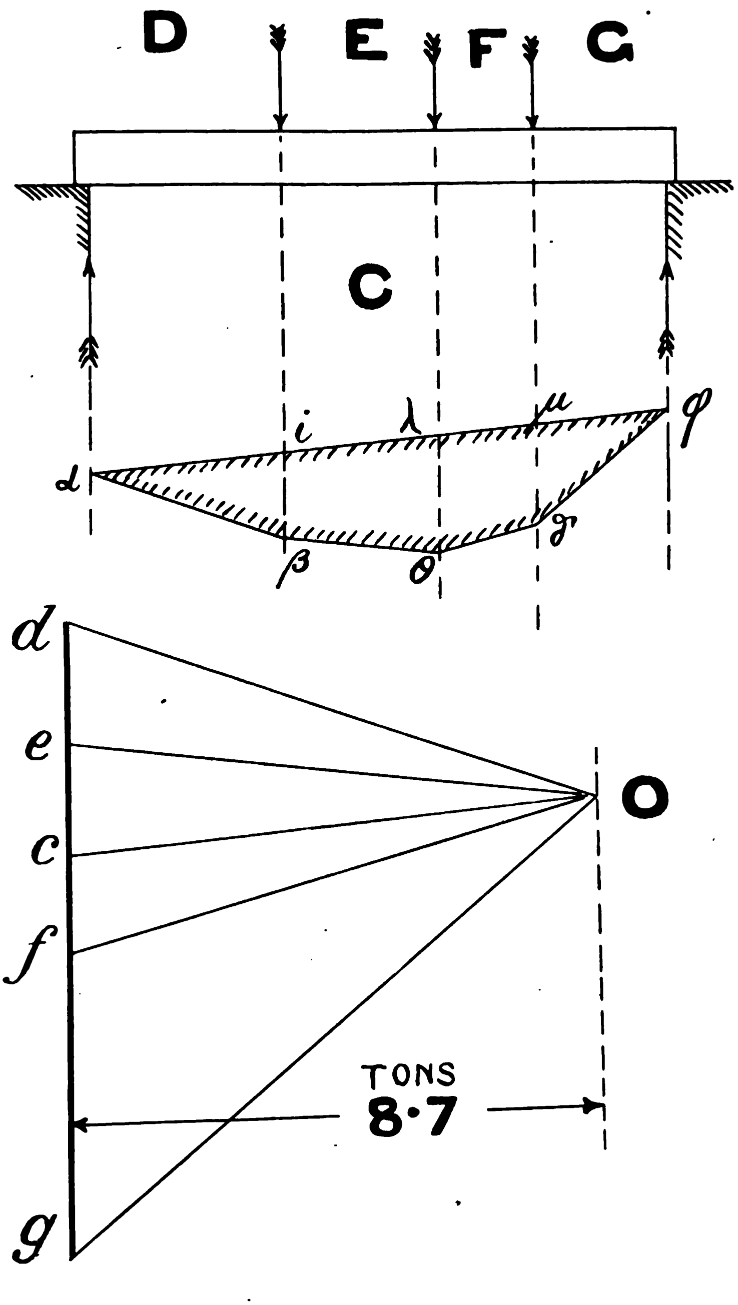
Member.	Stress in tons.	Member.	Stress in tons.	Member.	Stress in tons.
D 1	1·4	N 16	17·3	8, 9	3·4
D 3	2·3	P 14	14·7	9, 10	3·2
D 5	4·2	P 12	11·5	10, 11	4·2
E 7	6·7	P 10	9·0	11, 12	3·0
E 9	8·9	Q 8	6·8	12, 13	5·2
E 11	11·5	Q 6	4·2	13, 14	3·9
F 13	14·6	Q 4	2·4	14, 15	4·7
F 15	17·2	C 2	1·4	15, 16	4·7
F 17	19·7	C 1	1·7	16, 17	5·3
F 19	16·2	1, 2	0·8	17, 18	13·3
G 21	8·2	2, 3	1·2	18, 19	12·8
G 23	2·6	3, 4	1·7	19, 20	12·5
H 23	4·1	4, 5	2·6	20, 21	6·4
K 22	2·6	5, 6	1·5	21, 22	8·4
K 20	8·2	6, 7	3·7	22, 23	6·4
L 18	16·1	7, 8	2·5		

Each of the unknown loads comes out 3·2 tons, as measured on the stress diagram.

No. 65, 1891. **A beam of 30 ft. span is loaded, at distances of 7, 12, and 20 ft. from one end, with 5, 3½, and 2 tons respectively. Calling the points of application of the loads A, B, and C respectively, determine the moments about A, B, and C of the external forces acting on the beam on either side of those points.**

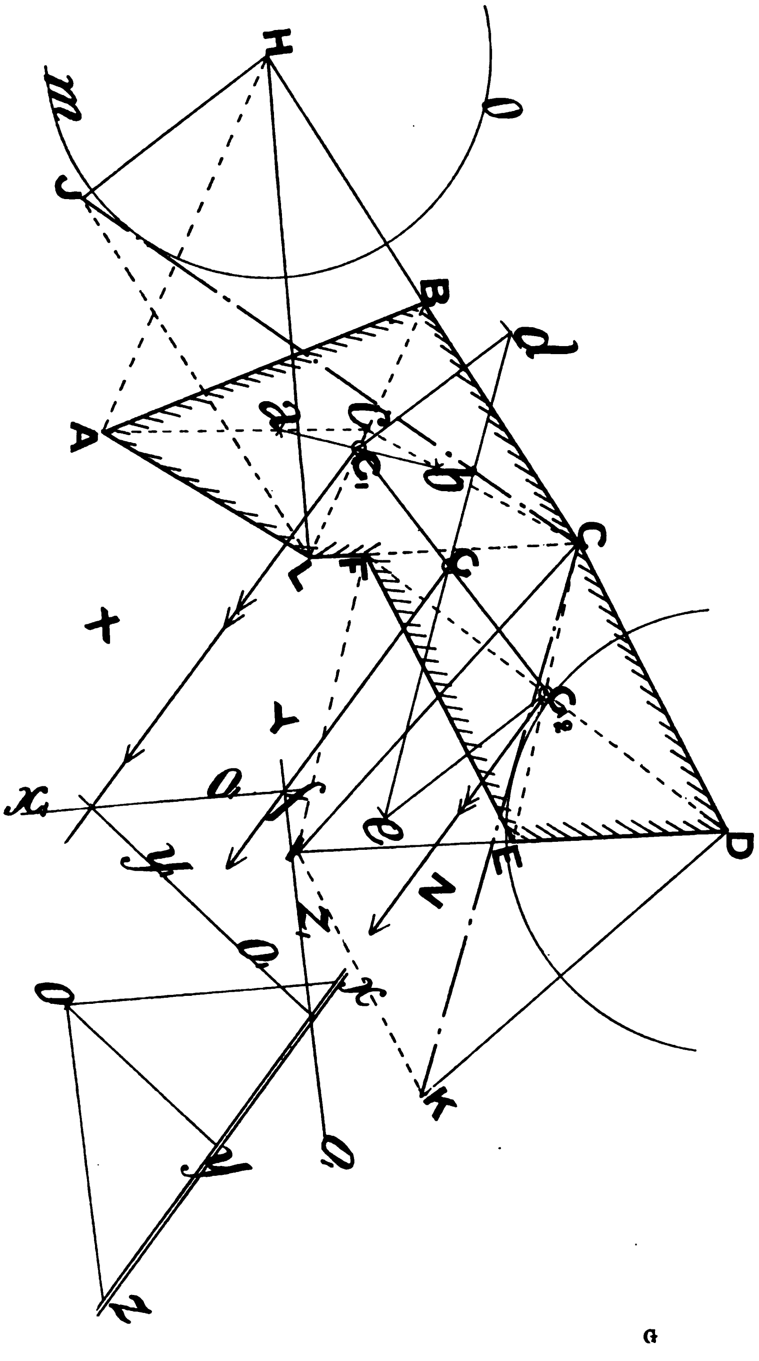
The beam is supposed supported at each end, and the distances in the question have been measured from the right-hand end. The moment of the external forces on one side of a point is the bending moment at that point. Set down the line of loads dg, after labelling the spaces between the forces (Bow's method). Take any pole, O, and draw the radial lines. Through any space, say E, draw $\beta\theta$ parallel to Oe, and through the space D draw $a\beta$ parallel to Od, and so on, until ϕ is obtained. Join $a\phi$, then the bending moment under DE = intercept $i\beta$ (full size) × horizontal pole distance of 8·7 tons = 4·5 ft. × 8·7 tons = 39·1 ton-ft. Similarly, the moment under EF is 53·7 ton-ft., and under FG is 45·5 ton-ft.

For full discussion of the theory of graphical determination of bending moment, consult Pullen's "Graphic Methods of Structural Design."


D
E
F
G
C
i
λ
μ
φ
α
β
θ
γ
d
e
c
f
g
O
TONS
8·7

No. 54, 1887. **Determine by construction the centre of gravity of the given figure ABCDEFL.**

It is generally found expedient to divide an irregular figure of this kind into triangles or parallelograms, as ABL, BCL, and CDEF, and determine the centre of gravity of each figure separately. Thus the centre of gravity of CDEF will be at G_2, the crossing-point of the diagonals; and the centre of gravity of the triangles ABL, BCL will be at *a* and *b*, one-third from *c* of the lines joining *c* to the vertices A and C of the triangles. Join *a* and *b*; the centre of gravity will fall on this line. The above process repeated for the triangles ACL, ACB, and their centres of gravity joined cuts the line *ab* in point G_1; hence G_1 is the centre of gravity of the two triangles combined. Draw the line G_1G_2; then the centre of gravity of the whole figure will fall on the line G_1G_2. The centre of gravity of the whole figure will be on this line at a point whose distance from G_1G_2 is inversely proportionate to the areas of the figures ABCL and CDEF. A line representing their areas will easily be found by reducing each to a triangle having the same altitude, the areas remaining the same. Thus draw AH parallel to LB, meeting CH produced in H; then area of △ CHL = area of Fig. CBAL. With H as centre, and any radius, describe arc *mo*; draw CJ tangential to the arc, and draw LJ parallel to the opposite side CH, meeting CJ in J. Then the △ CHJ = △ CHL (Euc. I. 37), having altitude H*m*, and CJ will represent the area; similarly CK will represent the area of Fig. CDEF. Consider these lines (or any fraction of them; $\frac{1}{3}$ taken in this case) to act through G_1G^2 as forces proportionate to the areas; then erect G_1d perpendicular to $G_1G_2 = \frac{1}{3}$CK, and $G_2e = \frac{1}{3}$CJ. Join *de*, cutting G_1G_2 in G; then the centre of gravity of the whole figure is at G, for the line G_1G_2 is divided inversely as the areas of the figures. Or the same result may be obtained by the aid of the funicular polygon. Thus, draw the forces


H
m
o
J
B
d
A
a
c
c1
b
C
L
F
c
X
Y
c2
x1
a
f
e
E
D
y
Z
N
a
x
o
K
y
o
z

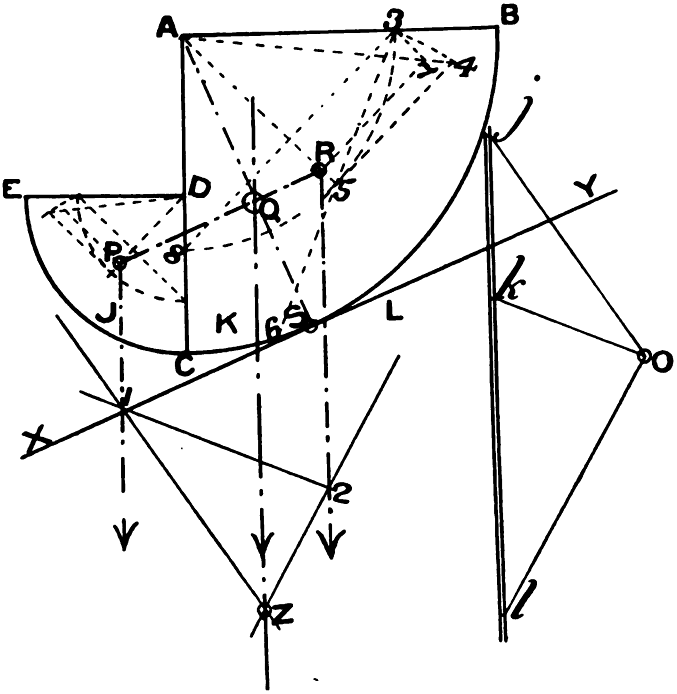
in any direction, as G_1x, etc.; draw the load-line parallel to the forces; make $xy = CJ$, and $yz = CK$; take any pole, o, and draw vectors xo, yo, zo; then the funicular polygon $x_1o_1y o_1 z o_1$ about the lines of action of the forces, thus x_1o_1 parallel to xo, y_1o_1 parallel to yo, z_1o_1 parallel to zo. Produce the first and last vector to meet in f; then f is a point in the line of the resultant of the two forces, and its direction is parallel to the forces, cutting line G_1G_2 in point G as before. This method is applicable when any number of forces have to be dealt with. Another way is to divide the figure into a number of parallel strips of equal width (the middle ordinate of each strip represents its area approximately), and apply the funicular polygon as above.

No. 64, 1889. **The given figure (two circular quadrants) is the elevation of a flat disc of uniform thickness. Determine the point on the curved surface on which the solid would rest on a horizontal plane.**

Label the given figure as ABC, DEC. It will be advisable to determine the centre of gravity of each quadrant separately, and then the centre of gravity of the two conjointly. First, then, let us take the larger segment ABC, and consider it as made up of an infinite number of small triangles having their apices at A; then the centre of gravity of each triangle will be in its centre line and one-third from the base. With A as centre, and radius two-thirds AB, draw the arc 3 5 8; then the centre of gravity of this arc will be the centre of gravity of the segment. Join A to 5, the centre point of the arc; join 3 5 and produce it to point 6, making 5 6 = 3 5. With 6 as centre, radius 6 3, describe arc 3 4, meeting the tangent through 5 in point 4, join A4, and draw 3 7 parallel to A5; through point 7 draw line 7R parallel to tangent 4 5, meeting A5 in point R: then R is the centre of gravity of the segment ABC.*

---

* For proof of this construction, consult question 64 of 1893.



(Note length of line 5 4, very nearly = arc 5 3.*) Proceed in exactly similar way to find P, the centre of gravity of the small segment DEC.

Now assume a force to act through P, and also through R, proportionate to the respective areas of the segments, which may be taken equal to the radius DE^2 at P, and AB^2 at R (areas of circles are as the squares on their diameters). Now draw the force polygon *jkl*, making $jk = DE^2$, $kl = AB^2$; select any pole, *o*, draw the vectors *jo*, *ko*, *lo*; parallel to these vectors draw the funicular polygon 1 2 Z, the first and last vector meeting in point Z: then Z is a point in the line of the resultant force, the direction of which must be parallel to the forces. Therefore draw ZQ parallel to the forces, meeting the line joining P to R in Q; then Q is the centre of gravity of the whole figure, and the resting-point must be at S, the shortest distance from Q to the circumference. Draw AQ, and produce it to meet the circumference at S. The horizontal plane is represented by *xy*, drawn through point S, and perpendicular to AS.

---

* See Clarke's "Graphic Statics."

No. 64, 1891. **Determine the centroid (centre of gravity) of the given figure.**

Let the given figure be ABCDEF. It is required to find its centre of gravity.

We may commence by dividing up the figure into triangles as shown. Next determine the centre of gravity of each triangle thus:—

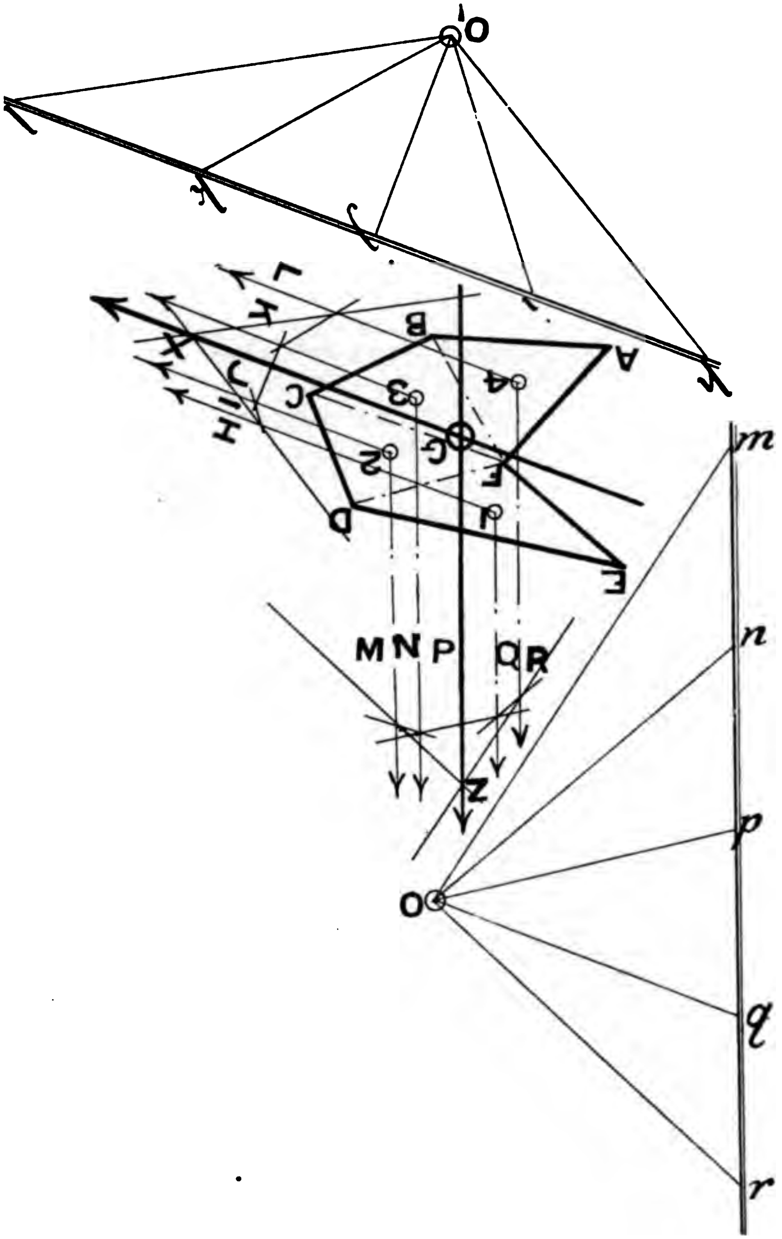
Centre of gravity of triangle		ABF	is at point	4
„	„	BCF	„	3
„	„	CDF	„	2
„	„	DEF	„	1

Now determine a line to represent the areas of the triangles, considering each as a force acting through the centre of gravity of each triangle.* The crossing point of the resultants of these forces, in two directions, will be the centre of gravity of the figure.

Consider the forces LK, KJ, JI, and IH acting in the directions shown. Draw a load-line, *hi*, parallel to the forces, making *hi* equal to the force acting through 1, and so on for each. Then draw the funicular polygon about the forces, the first and last vector meeting in point *x*. Then *x* is a point in the line of action of the resultant of the parallel forces. The resultant acting in the other direction through *z* is found in the same way. Their crossing-point, G, is the centre of gravity of the given figure.

---

* Construction fully discussed on page 80.


O'
A
B
C
D
E
F
G
H
I
J
K
L
X
1
2
3
4
M
N
P
Q
R
Z
O
m
n
p
q
r

No. 54, 1888. **A horizontal beam ab is so loaded that the vertical ordinates of the figure above it represent the intensity of the load at any point. Determine the point about which the beam will balance.**

Let ab be the given beam, $acdefgb$ the given figure above it representing the intensity of the load. Commence by dividing the figure into the triangles acb, bde, bfg, and reduce each of these triangles to triangles of equal altitude, the areas remaining the same as $1cb$, $bd2$, bfg * (note bg is taken as altitude of the last triangle); then their bases represent their areas. Consider forces acting through the centre of the triangles proportional to the bases of the triangles; then label the spaces between the forces A, B, C, and D. Now draw a line ad, Fig. 2, perpendicular to ab, equal to the three forces, making $ab = \mathrm{AB}$, $bc = \mathrm{BC}$, and $cd = \mathrm{CD}$, and, taking any pole Q, draw vectors $a\mathrm{Q}$, $b\mathrm{Q}$, $c\mathrm{Q}$, and $d\mathrm{Q}$. From any point in a vertical line through p draw a line $1'q$ parallel to the vector $a\mathrm{Q}$, cutting the direction of the force through p in $1'$. Draw $1'2'$ parallel to the vector $b\mathrm{Q}$, cutting next force in $2'$, and $2'3'$ parallel to the vector $c\mathrm{Q}$, cutting direction of force $t3'$ in $3'$, and finally $3'q$ parallel to the vector $d\mathrm{Q}$, intersecting $1'q$ in point q; then q is a point in the line of action of the resultant. Hence draw qr parallel to the forces, cutting the bar ab in point r; then r is the point about which the bar will balance.

---

* See question No. 54, 1887.

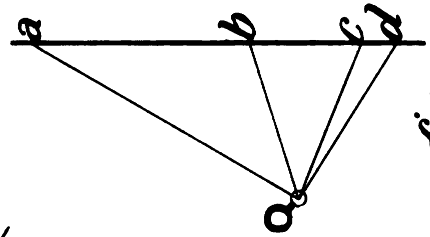
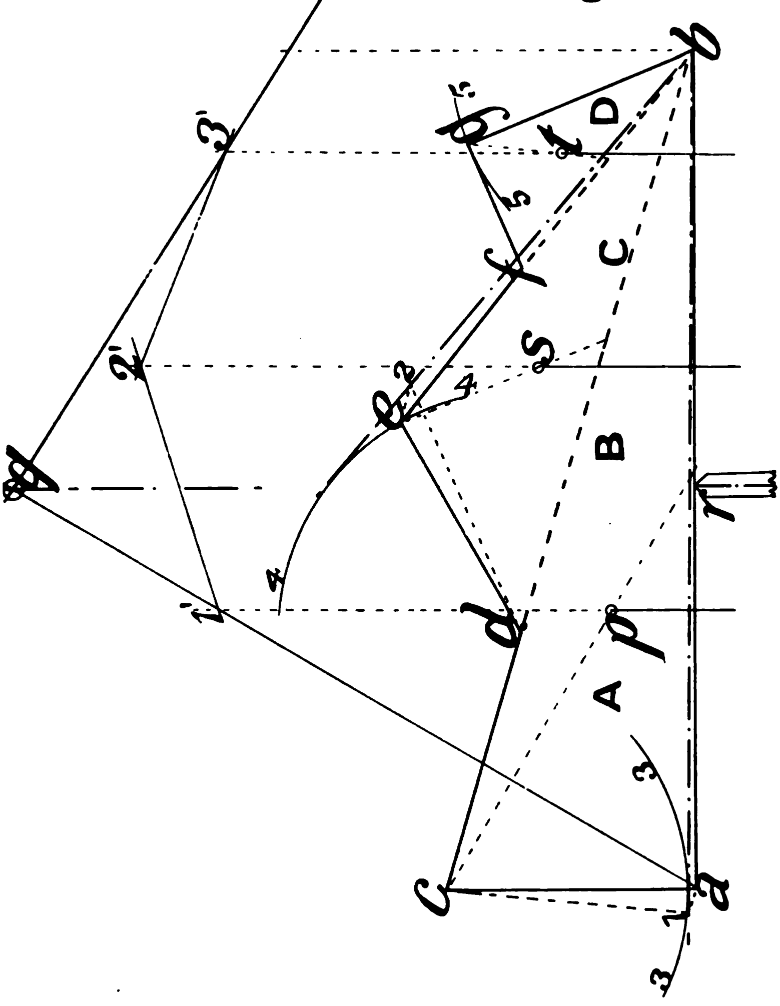


fig 2



No. 60, 1896. **The figure represents the section of a half-arch, ABDC. The span of the half-arch is 40 ft., rise 20 ft., thickness of key-stone 6 ft., thickness of arch-ring at abutment, AB, 9 ft.**

The loads on the half-arch are supposed concentrated on the verticals 1, 2, . . . 7, and are as follows:—

On vertical 1	**5·0 tons.**
„ „ 2	**5·3 „**
„ „ 3	**7·6 „**
„ „ 4	**6·0 „**
„ „ 5	**6·3 „**
„ „ 6	**6·6 „**
„ „ 7	**7·0 „**

The dotted circles *bd*, *ac* **represent the limits of safety, and are at one-third of the thickness of the arch-ring from the extrados and intrados respectively. The horizontal thrust (centre of pressure) at the crown is assumed to be along the line** *dt*, **and the curve of thrust is further assumed to pass through the point** *e* **where the vertical through the abutment VA meets the dotted curve** *ac*. **Determine and draw the curve of thrust between the points** *d* **and** *e* **by means of a funicular polygon, and give the value of the horizontal thrust at** *d*.

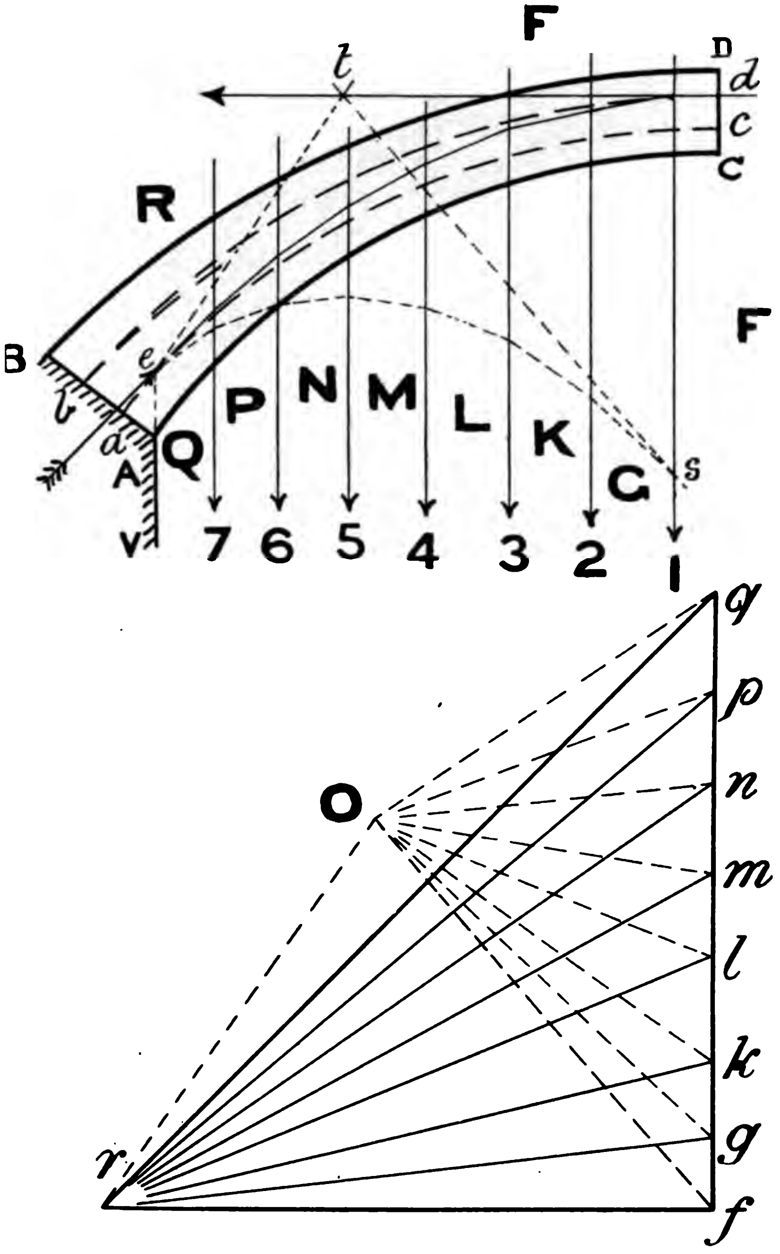
Scale of lengths, 10 ft. to an inch.

Scale of loads, 10 tons to an inch.

Set down the line of loads *qf* to the scale of 10 tons to the inch, in which the first load QP (= 7 tons) is represented by the vector *qp*, and so on for all the loads. Take *any* pole O, and draw the radial lines shown dotted. Then draw the funicular polygon [*] in the upper part of the figure (shown dotted), beginning with the point *e*: thus, through *e* and across the space Q draw a line parallel to O*q*, then through the point where this line cuts the line of action of the load QP draw a line parallel to O*p* across the space P, and so on, until the line parallel to O*g* cuts the line of action of GF in *s*.

---

[*] For a full discussion of the properties of the funicular polygon, consult "The Application of Graphic Methods to the Design of Structures," by W. W. F. Pullen, pp. 24, 38, and 254.


F
t
D
d
c
C
R
F
B
e
b
a
A
Q
P
N
M
L
K
G
s
V
7
6
5
4
3
2
1
q
p
O
n
m
l
k
g
r
f

Through *s* and across the space F (which lies between the lines of action of GF and FR) draw a line parallel to O*f*, cutting FR in *t*. Through *f* draw a line parallel to the horizontal thrust FR at the crown of the arch, and join *et*.

There must be a companion line in the pole diagram to *et* in the space diagram; hence through O draw a line parallel to *et*, which is the same as *r*O.

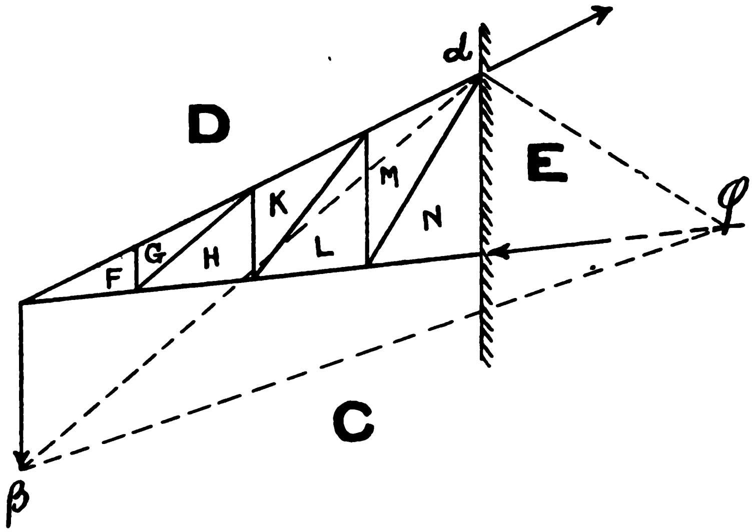
This construction determines the horizontal component *fr* to be 43·2 tons, and the new pole *r* for the line of resistance. Join *r* to each of the points *qpm*, etc., and across the spaces QPN, etc., draw another funicular polygon, beginning at the point *d* or the point *e*, and whose lines are parallel to *rq*, *rp*, *rn*, etc. This polygon will terminate at *e* or *d*, and will be the line of resistance under the given conditions.

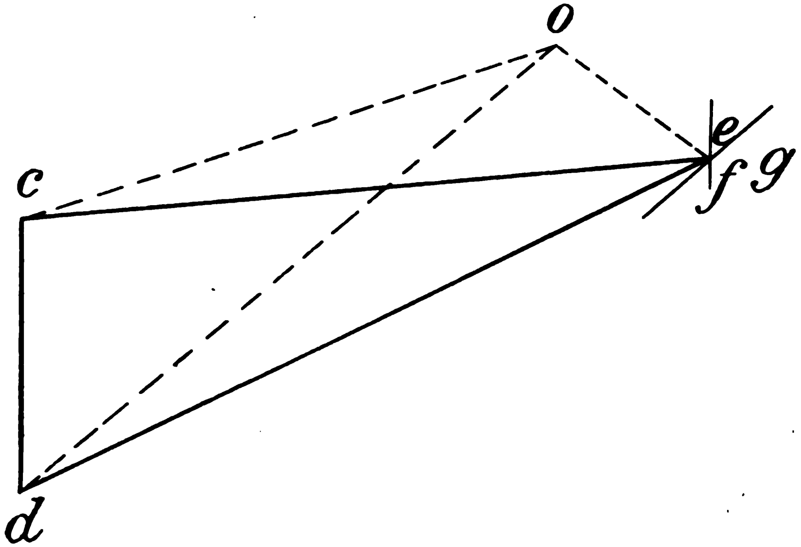
No. 65, 1890. **A braced cantilever of the given form supports a load of 1500 lbs. at its extremity. Determine and write down the stress in each of the bars forming the structure.**

In the original figure the frame in the upper part of the accompanying figure was given without the dotted lines, and without the two forces on the right. An assumption must be made as to the nature of the supporting forces on the right, before the problem can be started.

Let the lower force, CE, act in the direction of the lower member of the cantilever; the upper force, DE, must be found in magnitude and direction.

Draw the load-line *cd* to represent 1500 lbs. Take any pole, O, and draw O*c* and O*d*. Through α, and across the space D, draw $\alpha\beta$ parallel to *do*, cutting the line of action of the load in β. Through β, and across the space C, draw $\beta\phi$ parallel to *oc*, cutting the line of action of CE in ϕ. Join $\alpha\phi$, and through *o* draw *oe* parallel to $\alpha\phi$. Through C draw *ce* parallel to CE, cutting *oe* in *e*, and join *de*. To obtain the stress diagram: through *c* draw *cf* parallel to CF, and through *d* draw *df* parallel to DF. These lines exactly coincide with *ce* and *de*. Through *f* draw *fg* parallel to FG, and through *d* draw *dg* parallel to DG.





These lines intersect in *f*. Therefore *e*, *f*, and *g* are the same points. As *fg* is of no length, there can be no stress in FG. Similarly, it will be found that there is no stress in any of the vertical or diagonal bracing. The stress in the top boom of the cantilever is 4200 lbs., and that in the bottom boom is 3800 lbs.

No. 64, 1890. **State what graphic conditions must be fulfilled in order that a jointed polygonal frame may be maintained in equilibrium by a force acting at each joint. Illustrate your answer by a diagram.**

In order that it may be possible to solve a problem of this kind graphically, the following equation must hold :—

$$m = 2j - 3$$

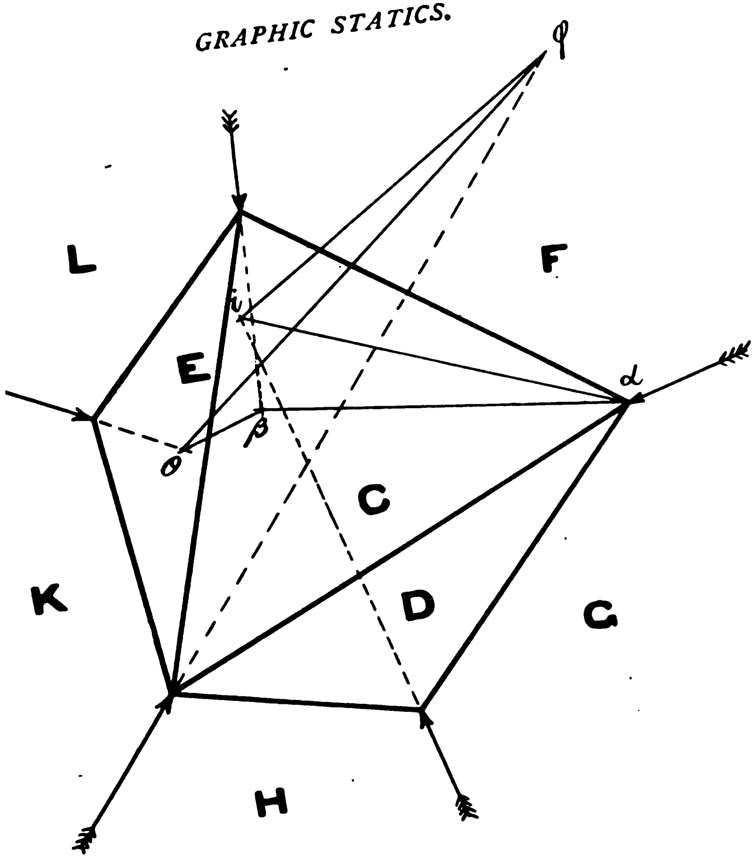
where m = number of members
and j = „ joints.

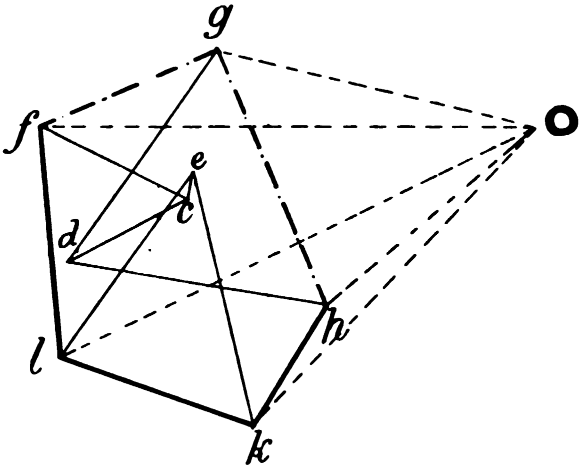
In the figure (upper part) the thick lines represent the members of the frame. There are five joints and seven members; therefore it is possible to find the stresses graphically, *i.e.* there are no superfluous members. The second condition is that the polygon of external forces must be a closed figure. Thirdly, the funicular polygon and the stress diagram must be closed figures. These conditions simply express the two laws of equilibrium graphically.

In the figure the magnitude of HG and the magnitude and direction of GF are unknown. It is required to find them to produce equilibrium. Draw the polygon of forces (line of loads) *flkh*, and the line *hg* of indefinite length; these are parallel to the corresponding forces they represent. Take any pole, O, and draw the radial lines. Beginning at the point a,* draw $a\beta$ through the space F, and parallel to the line O*f*. Through β draw $\beta\theta$ across the space L and parallel to O*l*. Then through θ draw $\theta\phi$ across the space K and parallel to O*k*; and through ϕ draw ϕi, across the space H, and parallel to O*h*, cutting HG produced in *i*. Join *ai*, and through O draw O*g* parallel to *ai*, cutting *hg* in *g*. Join *fg*, then *fg* is the magnitude and direction of FG, and *hg* is the magnitude of HG. The stress diagram is then drawn.

---

* See Pullen's "Graphic Methods of Structural Design," p. 37.


φ
L
F
ι
E
α
β
θ
C
K
D
G
H


g
f
O
e
c
d
h
l
k

No. 60, 1895. **A horizontal beam (shown in upper part of figure) rests on two supports at its extremities, and is loaded as shown. The magnitude of the three left-hand loads is 100 lbs. each, and the other two 120 lbs. each.**

(1) Find the moment of the resultant, relatively to the line *mn* of the forces acting to the left of *mn*.

(2) Considering the loads as forces, find their moment of inertia (in foot-pounds) relatively to the line *mn* and the corresponding radius of gyration. Unit of weights, 0·1 in. = 20 lbs.; linear unit; $\frac{1}{8}$ in. = 1 ft. In constructing the funicular polygon, take 16 ft. as base length.

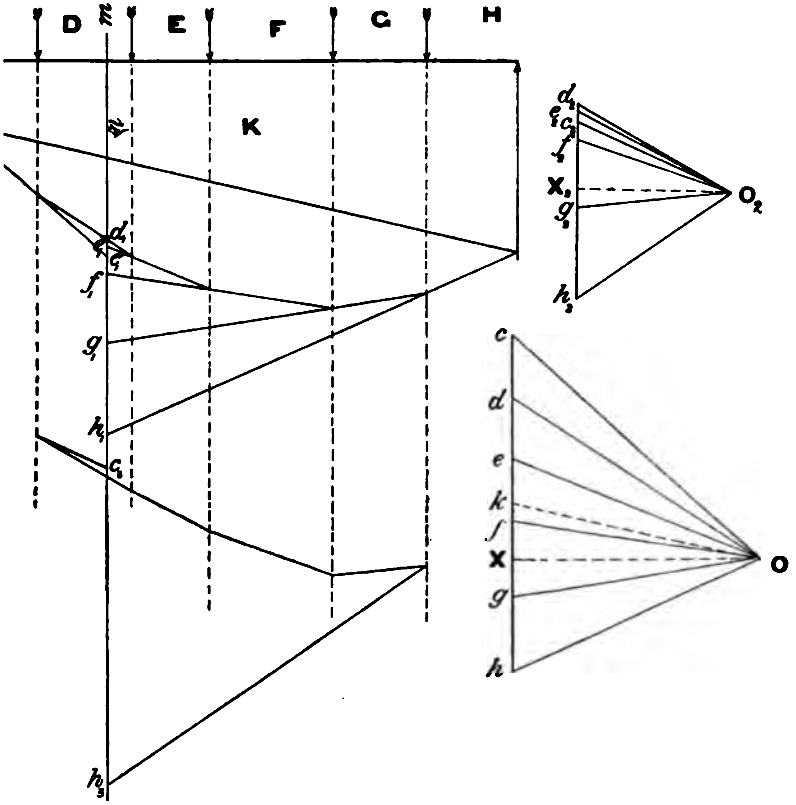
Label the spaces between the forces as in Bow's method, and set down the line of loads *chl*. Draw the funicular polygon with sides parallel to the radial lines *oc, od,* etc. Draw the line *ok* parallel to the closing line of the funicular polygon; then *ck* represents the supporting force on the left, and *hk* that on the right. OX was made to represent 400 lbs.

Each side of the funicular polygon has been produced to cut the line *mn* in c_1, d_1, etc. The moment of the resultant of all the forces on the left of *mn* about *mn* = sum of the moments of all the forces to the left of *mn* about *mn* * = real length represented by nd_1 × force represented by OX = 5·25 × 400 = 2100 foot-lbs.

The line of intercepts d_1, e_1, c_1, etc., has been reproduced on the right of the figure at d_2, e_2, c_2, etc., to avoid confusion. Take any pole o_2 and draw radial lines to the points d_2, e_2, etc. The line O_2X_2, being in the diagram of lengths, is made to represent 10 ft.

---

* See "Graphic Methods of Structural Design," by W. W. F. Pullen, p. 24.


D
m
E
F
G
H
K
n
d₁
e₁
c₁
f₁
g₁
h₁
c₃
h₃
d₂
e₂
c₂
f₂
X₂
g₂
h₂
O₂
c
d
e
k
f
X
g
h
O

Now draw the second funicular* polygon with sides parallel to o_2, d_2, etc., beginning at c_3 and ending at h_3. The line c_3h_3 represents a length, because it is in the diagram of lengths. It represents 20·5 ft.

$$\begin{aligned}\text{The moment of inertia} &= c_3h_3 \times OX \times O_2X_2 \\ &= 20{\cdot}5 \times 400 \times 10 \\ &= 82{,}000\end{aligned}$$

If $M =$ total mass, and $R =$ the radius of gyration of the mass, then—

$$\text{Moment of inertia} = MR^2$$

$$\begin{aligned}\text{In this case } M &= 100 + 100 + 100 + 120 + 120 \\ &= 540 \text{ lbs.} \\ \therefore 82{,}000 &= 540 \times R^2 \\ R^2 &= 151 \\ \text{and } R &= 12{\cdot}3 \text{ ft.}\end{aligned}$$

The square root may be performed geometrically by Euc. III. 36.

In any case it is waste of time to find the moment of inertia geometrically, as it can be done much quicker arithmetically by means of a slide-rule or pocket calculator.

NOTE.—Read note at end of No. 61 of 1896, referring to base line or horizontal pole distance. It should also be noted that the moment of inertia cannot be expressed in *foot-pounds* (it is feet^2 × pounds). The examiner should have said in feet and pounds units, meaning that lengths are all measured in feet, and masses in pounds.

---

* See "Graphic Methods of Structural Design," by W. W. F. Pullen, p. 203.

A horizontal beam 20 ft. long is loaded as shown, at equal intervals. Each weight is 1 ton.

(1) The beam supposed *resting freely* on the two supports, one at each end. Determine, by means of the funicular polygon, the shearing stress (in tons) and the bending moment (in foot-tons) at sections under the loads DE and NP.

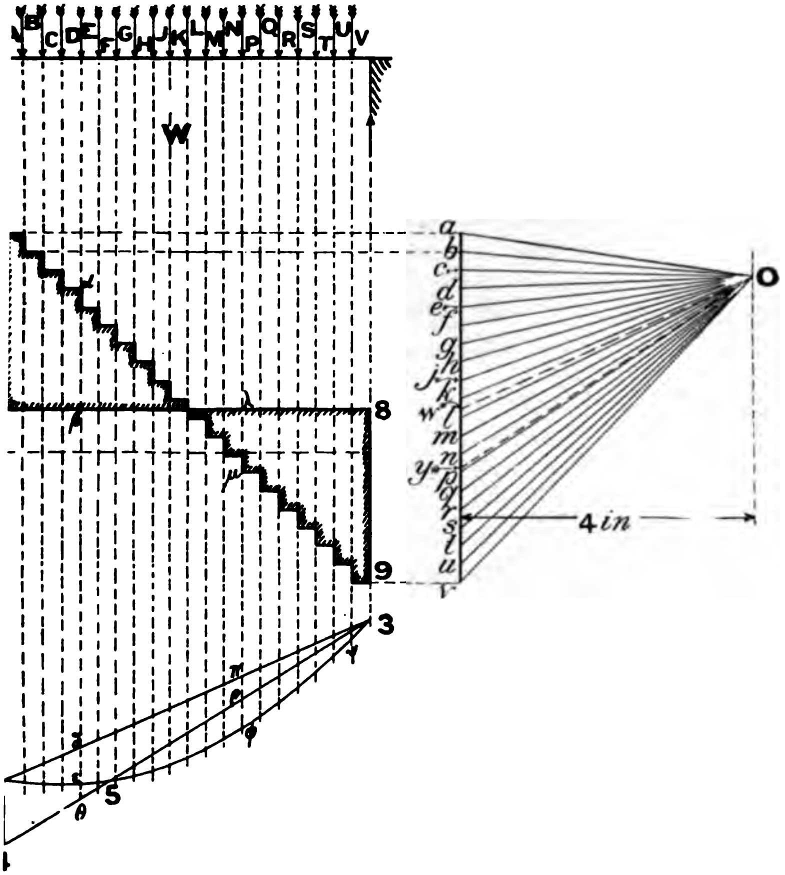
(2) The beam supposed *embedded firmly* at the left-hand end. Assuming what actually occurs in this case, that three-eighths of the total load is supported at the right-hand end, show how the measurements determined by the funicular polygon in case 1 can be modified by an additional line, so as to obtain the stresses in this case. Determine the bending moment (in foot-tons) at the left hand end and under the load NP. Scale of lengths, 4 ft. to one in.; scale of loads, 4 tons to 1 in. Employ for polar distance, 20 ft.

Taking the first case, in which the beam is only supported at the ends, put down the line of loads *av* (shown on the right of the figure), mark off a distance to the right of 4 in. and draw a line parallel to the line of loads. Take a pole, O, anywhere in this vertical line, and draw the radiating lines O*a*, *ob*, etc. Produce the line of action of each load as far as required, and through *a*, *b*, etc., draw horizontal lines (shown dotted) which shall also pass through the spaces A, B, etc., cutting the lines of action on each side of a space. In this way the shearing-force diagram 7 6 9 8 is drawn. The shearing force at any section is the sum of the forces on one side of that section. The line 7 8 is drawn right across the diagram, and if produced will pass through the point *w*. The supporting forces are *vw* and *wa*.

To obtain the bending moment diagram draw through the space V a line parallel to O*v* (in the right-hand diagram). The point 3 can be taken anywhere in V.W. Through the end ψ of the parallel line just drawn, draw another line parallel to O*u* and through the space U. Continuing in this way, the point 2 is reached. Join 2 3, and through O draw O*w* parallel to 2 3. Then *w* divides the line of loads into the two reactions (in this case equal). The figure 3 2 5 is the bending moment diagram for case 1. The moment under DE is represented by $\epsilon\delta$, and its value equals $\epsilon\delta$ in inches × scale of lengths × the pole distance (4 inches) × the scale of the pole diagram (4 tons to 1 inch) = 32 foot-tons. Under NP the bending moment will be $\pi\phi$ in inches × scale of lengths (4 feet to 1 inch) × pole distance (4 inches) × scale of loads (4 tons to 1 inch) = 44·8 foot-tons. The shearing force on the right of NP is $\lambda\mu$ in inches × scale of loads = 3·5 tons, and on the left it is 2·5 tons. The shearing force to the left of DE is $\alpha\beta$ × scale = 6·5 tons, that to the right being 5·5 tons.

When the beam is embedded at its left-hand end, the right-hand reaction is $\frac{3}{8}$ of 19 tons = $7\frac{1}{8}$ tons; therefore set off $vy = 7\frac{1}{8}$ tons. Join Oy (shown dotted), and through 3 draw 3 4, cutting the parabola 3 2 in 5. The vertical ordinate of the triangle 2 3 4 represents the moment at that section that is put on the beam in the opposite direction on account of fixing the left-hand end; and it is of opposite sign to that of the unfixed beam. The resultant moment when the beam is fixed is therefore given by the difference of the ordinates of the parabola and triangle. The moment at the section under NP = the ordinate $\rho\phi$ in inches × scale of lengths × pole distance × scale of loads = 25·6 foot-tons, and is of the same sign as the previous moments. Similarly, the moment under DE = full-size length of $\epsilon\theta$ × number of tons represented by the horizontal pole distance = −12·8 foot tons, the negative sign indicates a tendency to bend in the opposite direction.

It is wrong to say "Employ for pole distance 20 feet," as in the question. The pole diagram contains the loads, and is a *diagram of forces* only, and no line in it can represent a length. The left-hand diagram is a *diagram of lengths*, and cannot represent forces. Hence the pole distance cannot represent 20 feet. It has been taken as 4 inches, and represents 4 × 4 = 16 tons. For a full discussion of the graphical determination of Bending Moment consult Pullen's "Graphic Methods of Structural Design."


A
B
C
D
E
F
G
H
J
K
L
M
N
P
Q
R
S
T
U
V
W
8
9
3
5
1
θ
π
φ
μ
λ
a
b
c
d
e
f
g
h
j
k
l
w
m
n
y
p
q
r
s
t
u
v
O
4 in

No. 64, 1893. **Find the centre of gravity of the plane surface ABBFCD.**

The figure is symmetrical with respect to FE, hence its centre of gravity must lie in that line.

The figure ABFCD = EBFC – EAD.

Let the centre of gravity of ABFCD be at G_3, and
" " " EBFC " G_1, and
" " " EAD " G_2.

We may consider weights proportional to the areas of the figures to act at their centres of gravity, then the point about which the sum of the moments of weights is zero is the centre of gravity,* remembering that the area EAD is negative, and consequently the weight proportional to that area will also be negative. Take moments about the point G_3; then, by the above reasoning—

$$\text{Area of EBFC} \times G_1G_3 - \text{area of AED} \times G_2G_3 - \text{area of ABFCD} \times 0 = 0$$

$$\text{or area of EBFC} \times G_1G_3 = \text{area of AED} \times G_2G_3$$

$$\text{and } \frac{\text{area EBFC}}{\text{area AED}} = \frac{G_2G_3}{G_1G_3}$$

$$\text{Now, the area of EBFC} = \frac{\text{arc BC} \times \text{BE}}{2} = \text{arc BF} \times \text{BE}$$

$$\text{and the area of AED} = \frac{2\ \text{AE} \times xy}{2} = \text{AE} \times xy$$

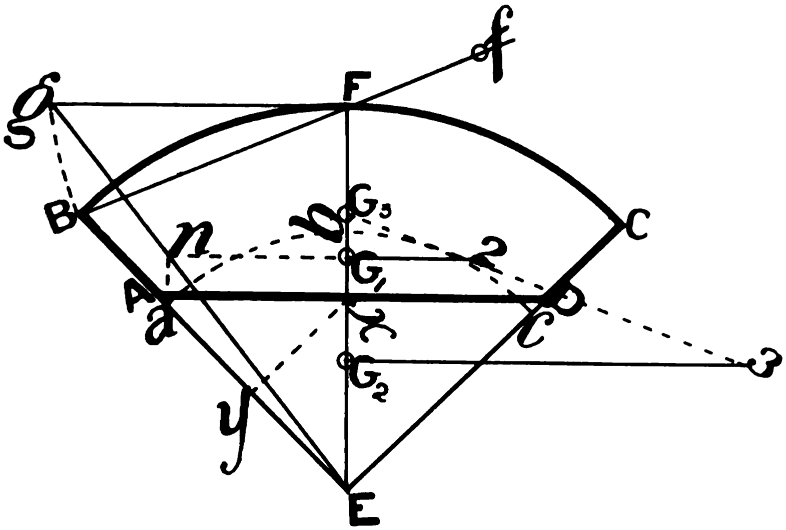
$$\text{therefore } \frac{G_2G_3}{G_1G_3} = \frac{\text{arc BF} \times \text{BE}}{\text{AE} \times xy}$$

Hence when G_1 and G_2 are known, set up $G_2 3$ and $G_1 2$ proportional to the products, arc BF × BE and AE × xy parallel to one another. Join 3 2, and produce it to cut EF in G_3, which is the required centre of gravity.

The centre of gravity of AED is at G_2, where $G_2 x = \frac{xE}{3}$.

---

* *Vide* Minchin's "Statics," p. 227; or Pullen's "Graphic Methods of Structural Design," p. 37; or any treatise on Mechanics.



The area of the sector EBFC is made up of a large number of small triangles, the centres of gravity of which will lie on a circular arc, *abc*, whose radius E$a = \frac{2}{3}$ EB. Hence the centre of gravity of the sector is the same as the centre of gravity of the arc *abc*. To find this point, set off along the tangent at F, the length Fg = arc BF. This may be done by Rankine's method, or simply stepped off with a small compass. In this case BF has been joined and produced to f such that $Ff = \frac{BF}{2}$. With centre f the arc Bg has been drawn, then Fg = arc BF. Join gE, and through a draw an parallel to EF, cutting gE in n; and then draw nG_1 parallel to gF, cutting FE in G_1.* Then G_1 is the centre of gravity of the sector. That G_1 is the centre of gravity may be shown thus: By the theorem of Pappus,* generally ascribed to Guldinus, if the curve *abc* is rotated about an axis through E perpendicular to EF, then area of the surface traced out by the curve *abc* during one revolution equals the length of the path of the centre of gravity $G_1 \times$ the length of curve *abc*. The area traced out by *abc* is in this case a zone of a sphere, and consequently equals the curved area of the circumscribing cylinder,† which equals $2\pi \times \mathrm{E}b \times \text{chord } ac = 2\pi \times ea \times 2n\mathrm{G}_1$. Therefore—

---

* *Vide* Clark's "Graphic Statics."

$$2\pi \times ea \times 2n\mathrm{G}_1 = 2\pi \times \mathrm{EG}_1 \times \text{arc } abc$$

and, as arc $abc = \frac{2}{3}$ arc BFC, and $ea = \frac{2}{3}\mathrm{EB} = \frac{2}{3}\mathrm{EF}$, the above equation becomes—

$$\frac{2}{3}\mathrm{EF} \times 2n\mathrm{G}_1 = \mathrm{EG}_1 \times \frac{2}{3}\mathrm{BFC} = \mathrm{EG}_1 \times \frac{2}{3} \times 2 \text{ arc BF}$$

$$\text{or } \mathrm{EF} \times n\,\mathrm{G}_1 = \mathrm{EG}_1 \times g\mathrm{F}$$

$$\textit{i.e. } \frac{n\mathrm{G}_1}{g\mathrm{F}} = \frac{\mathrm{EG}_1}{\mathrm{EF}}$$

---

* *Vide* Minchin's "Statics," p. 271; or Goodeves' "Principles of Mechanics," p. 160; or Edwards' "Elements of Geometry."

† *Vide* Wilson's "Solid Geometry," p. 58.

CHAPTER III.

SOLID GEOMETRY—LOCI.

OP is the horizontal trace of a plane inclined at 30°; Q and X are two points in the plane. (*a*) A particle is discharged from P, along the line PR in the plane, with a real velocity *u*. Find and draw the curve of the path so that the particle will pass through Q, and determine and write down the value of *u*. (*b*) With *u* thus determined, find the direction of the discharge from P, so that the path of the particle shall pass through X; and draw the curve of the path. How many solutions are there?

The end elevation of the plane is shown at O*b*, and the angle *b*OM is 30°. PR is the plan of the line in the given plane along which the particle is projected, and consequently it is a tangent to the plan of the path of the particle at P, the point of projection. To draw the path from P to Q (which will be a parabola), two points on the curve are given, viz. P and Q, and a tangent to the curve at the point P. Join PQ, and bisect it in E. Through E draw a line perpendicular to OP, cutting PRV in V. Join VQ. Then the middle point T of EV will be a point on the curve.* Also STR, a line through T parallel to QP, is a tangent at T. Join QT, and through S draw SKU parallel to VE. The middle point K of SU is a point on the curve. In the same way join QK, and through K draw the tangent WKJ parallel to QT. The middle point of the line WN (drawn through W parallel to VE) is a point on the curve. In this way any number of points can be found, and the complete parabola PTKQ can be drawn.

Next, to find the velocity of projection *u* from the point P. The *true* angle RPO must first be found. Take any point *k* in PV. Its elevation is *b*; on the left of the figure. Rotate *b* about O down to M, and thence in plan to *h*.

---

* Consult Wilson's "Solid Geometry and Conic Sections," p. 104.

The angle hPO measures 58°, and $\tan 58° = 1\cdot6$, which is found by construction.

If u be the velocity of projection along the plane, $u \cos 58° =$ the horizontal component of the velocity.

The acceleration down the plane due to gravity is $g \sin 30°$, and consequently the component of the particle's motion up the plane perpendicular to OP is $tu \sin 58° - \frac{1}{2} gt^2 \sin 30°$ at the end of t seconds. At the same instant the horizontal component of the motion parallel to FP is $tu \cos 58°$. Dividing one by the other, we get—

$$\frac{\text{component up the plane}}{\text{horizontal component}} = \frac{tu \sin 58° - \frac{1}{2}gt^2 \sin 30°}{tu \cos 58°}$$

Let the component up the plane be denoted by y, and the horizontal component parallel to $\text{FP} = x$, then the above equation reduces to—

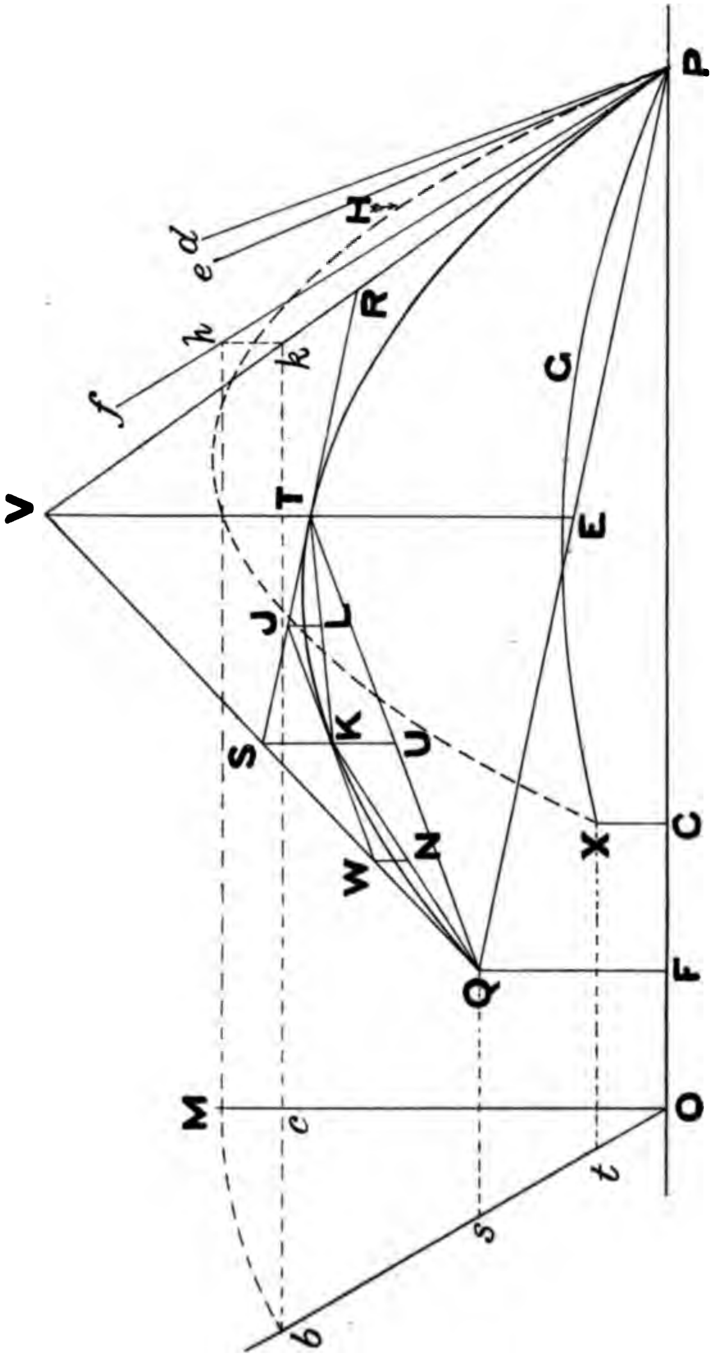
$$x \tan 58° - \frac{x^2 g \sin 30°}{2u^2}(1 + \tan^2 58°) = y$$

It should be remembered that $\frac{1}{\cos^2 58°} = \sec^2 58° = 1 + \tan^2 58°$, and that $x = tu \cos 58°$.

The above equation is satisfied for all points on the curve, and therefore for the point Q. Inserting $y =$ real length of $\text{QF} = \text{SO} = 1\cdot1$ in., and $x = \text{FP} = 4\cdot6$ in., together with $\sin 30° = \frac{1}{2}$ and $\tan 58° = 1\cdot6$, we find $u = 9\cdot8$ in. per second.

Now, with this value of u, and $x = \text{CP} = 3\cdot84$ in., also $y =$ real length of $\text{XC} = \text{O}t = 0\cdot4$ in., solve the above equation for $\tan a$, where a is the real inclination of the new line of projection, with OP. In this way we obtain $\tan a = 2\cdot66$ and $0\cdot5$, two real values, showing that there are two directions along which the particle can be projected with a velocity u so as to pass through X. The parabola PGX is the plan of the one path, while the dotted parabola PHX is the plan of the other path.

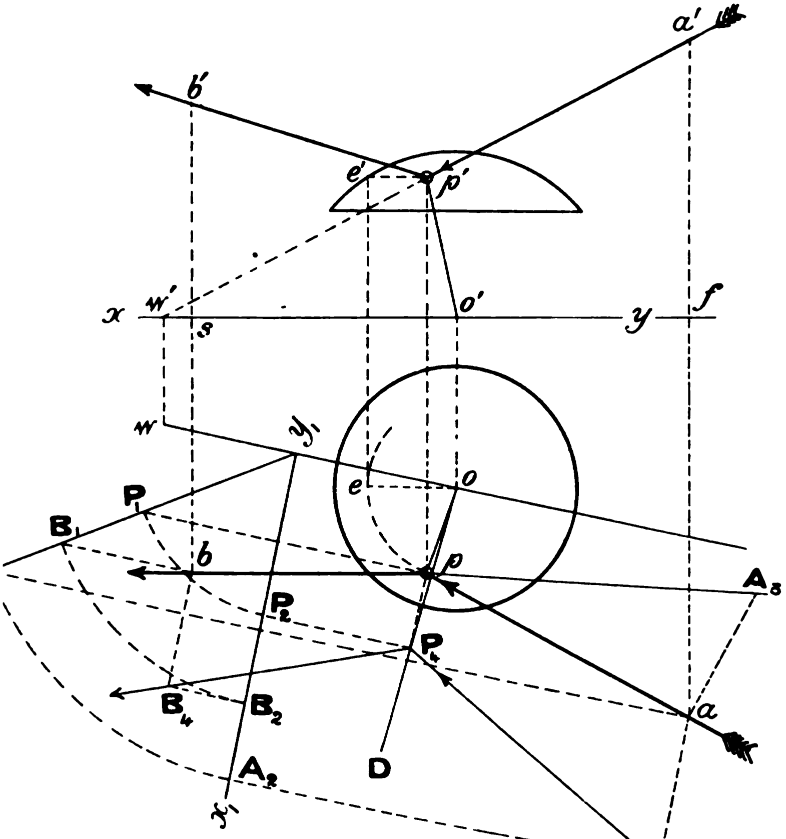
It must be remembered that the *plans* of the angles whose tangents are $2\cdot66$ and $0\cdot5$ must be obtained before the parabolas are drawn.


V
P
d
e
H
R
h
k
f
T
E
G
J
L
K
U
S
N
W
Q
F
C
X
M
c
O
t
s
b

No. 60, 1892. **The figure represents the plan and elevation of a spherical gun shield. A projectile, whose line of flight is inclined downwards at an angle of 25°, strikes the shield. The plans of the line of flight (*pa*) and the point of impact (*p*) are given. Assuming the projectile to be deflected, after impact, at an angle to the normal at *p* equal to that at which it arrives, and to move forward in the plane containing *pq* and the normal, determine its path after impact (unit 0·1 in.).**

The centre of curvature of the spherical surface is at O in plan, and *o′* in elevation, while the radius of curvature at the point of impact *pp′* is *op*, *o′p′*. The path along which the shell will rebound from the curved surface will lie in a plane containing the incident path and the radius of curvature at the point of impact. First obtain the elevation of the incident path. Take any point *a* in the plan of the incident path, and at *a* erect aA_3 perpendicular to *pa*. At *p* set off the $\angle\ apA_3 = 25°$; then the point A is actually situated at a height $= aA_3$ above the level of P. Above *a* set up $fa' = aA_3 +$ height of P above ground. Then *a′p′* is the elevation of the incident path.

The horizontal trace of the plane containing AP and OP will be *wo*, *w* and *o* being the horizontal traces of AP and OP. Rotate that plane about *wo* down to the ground, and the reflected path can immediately be drawn in. Thus, make an end elevation of the plane on x_1y_1 as new groundline, which is perpendicular to *wo*. Set up the point P_1 above *p* at its given height above the ground. Join P_1y_1. The new elevation of A will be at A_1. When the plane is rotated down to the ground the points A_1, B_1, and P_1 come down to A_2, B_2, and P_2. The actual points A and P will come down to A_4 and P_4, and OD is the actual normal at the point P. Set off $\angle\ B_4P_4D = \angle\ DP_4A_4$; then P_4B_4 is the actual reflected path. Rotate the plane back into its former position, and the point B_4 falls back to *b* in plan and *b′* in elevation. Therefore *pb*, *p′b′* are the projections of the reflected path as the shell leaves the point P.

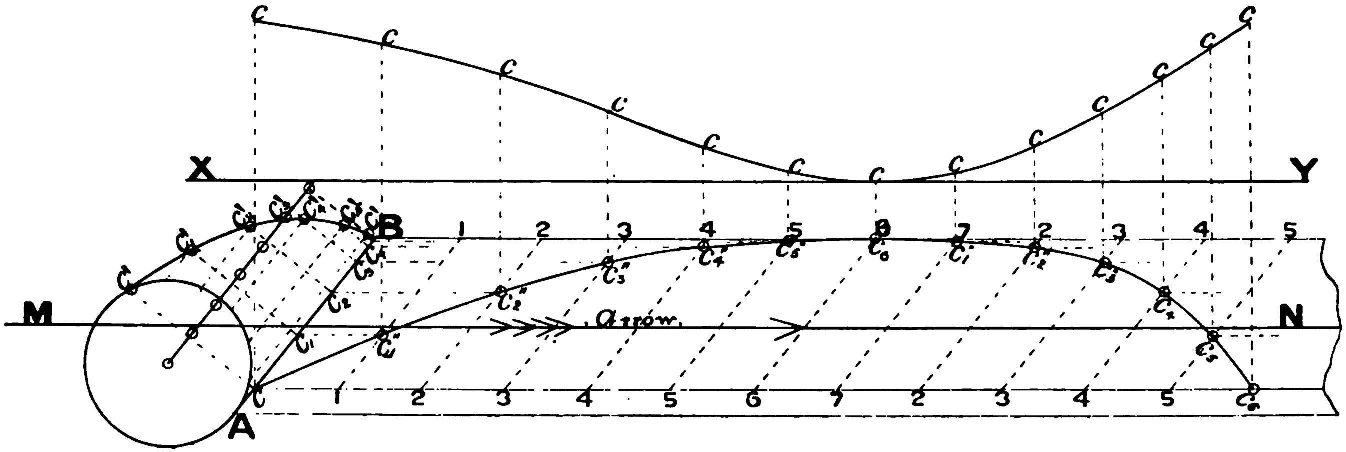

a'
b'
e'
p'
x
w'
s
o'
y
f
w
y₁
e
o
P₁
B₁
b
p
A₃
P₂
P₄
a
B₄
B₂
A₂
D
x₁
A₄

No. 47, 1888. **AB is a line drawn on the floor of a railway truck moving at a speed of 5 miles per hour in the direction indicated by the arrow. A wheel whose diameter is 14 ft. is rolled across from A to B and back in 12 seconds. Trace the plan of the locus of a point on the circumference of the wheel, and make an elevation of this locus on a plane parallel to the direction of motion of truck (scale $\frac{1}{8}$ in. = 1 ft.).**

The line AB and the arrow were given only. Now, as the truck moves in the direction of the arrow, draw from A and B lines parallel to the arrow, which will represent the extent to which the wheel is rolled in all positions, and as the truck moves at the rate of 5 miles per hour, it will travel in one second $\frac{5 \times 1760 \times 3 \text{ ft.}}{60 \times 60} = 7\frac{1}{3}$ ft. Therefore set off distances 1, 2, etc. = $7\frac{1}{3}$ ft., and through every division thus found draw lines parallel to AB, which will represent the position of wheel in plan at the end of every second when moving in the direction of the arrow. Make an elevation of the wheel on AB as shown, and measure off on the circumference of the wheel a length of arc AC′ = AB, and divide it into six equal parts to represent the position of the wheel centre at the end of every interval. As the wheel is rolled over from A to B, the point C will trace a cycloidal curve as shown;* then in one second the wheel will have rolled over one-sixth of AB, and point c' will have moved to c_1. Determine its plan c'. At the same time the wheel will have travelled in the direction of the arrow to position 1 1, therefore join c_1' to c_1''; then c_1'' will be a point in the path of the locus. All other points are found in exactly the same way, the points thus found, being joined freehand, will give the complete locus. Notice that on the wheel's backward journey from B to A, the point moves backwards, consequently the curve is much shorter. The elevation is easily found by taking xy parallel to the arrow, and erecting perpendiculars from each point of locus. The heights will be readily obtained from the cycloidal curve; thus height of $c_1'' = c_1'c' \cdot c_2'' = c_2'c^2$, etc.

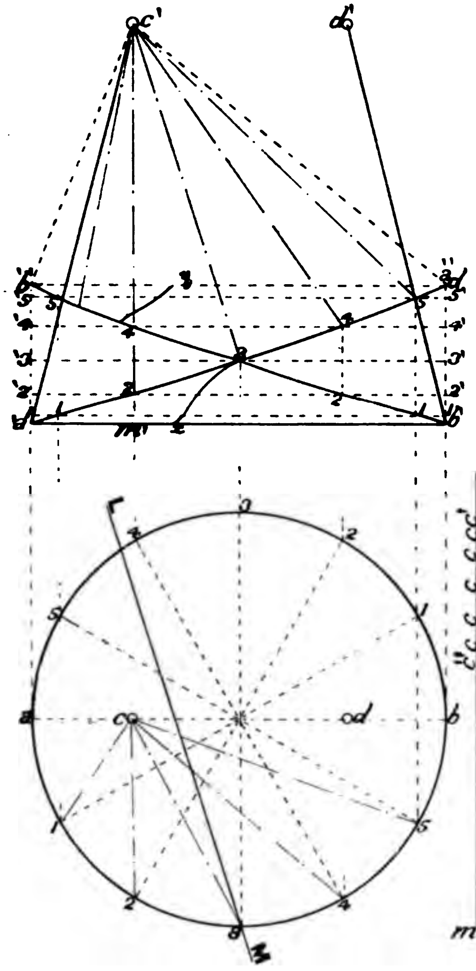
---

* Angel's "Geometry," p. 95.


c
X
Y
M
N
A
B
Arrow
1
2
3
4
5
6
7

No. 46, **A bar AB is suspended from two points, C, D, by two equal strings, AC, BD, and is turned horizontally through two right angles so that it comes into position shown by dotted lines a'', b' on the elevation. Draw the elevation of the locus of the ends of the bar, and make a section on the line LM of the surface it generates.**

The length of the strings and the length of the bar *ab* will in all positions remain constant, while the length of plan of string *c*1*c*2*c*3, etc., will vary as the bar is turned. This variation determines the amount the bar will have risen in all positions, and the ends of the bar will describe a circle in plan. Divide this circle into any number of equal parts, *a*1 . . . 12, etc. Now, as *ca* will have turned to *c*1, it will be noticed that end *a* of the bar will have risen. This amount will easily be found by considering the string a perpendicular from *c* to the bar, and a portion of the bar as forming a right-angle triangle. Draw *mb*, Fig. 2, making $ca = ma$, $m1_1 = c1$, $m2, = c2$, $m_1 3_1 = c3$, $m_1 4_1 = c4$, $m_1 5_1 = c5$. At *m* erect a perpendicular to *m*6, and with *a* as centre and radius equal to the length of string $c'a'$, find points c', and with 1, 2, 3, 4, 5 as centres respectively cut off 1*c*, 2*c*, 3*c*, 4*c*, 5*c*, $bc = c'a'$. The bar will have risen from c' to c'' while being turned through two right angles. Take these lengths and transfer to the elevation, such as a $1'$, $2'$, $3'$, $4'$, $5'$, b'', placing *c*1 at *a*, c'' at b'', through each of these divisions draw horizontal lines as shown, these being the levels of the ends of the bar at different stages; erect a perpendicular from point 1 to 1_1, and from point 2 to 2_1, etc. The curve drawn through these will be the path of the end a'; the same at the back will give the locus of the end *b*. The section is easily obtained by erecting verticals from the points in plan were the plane LM cuts the bar at different stages, and is shown by the curve 4*y* at the back, and the curve *x*3 in front.



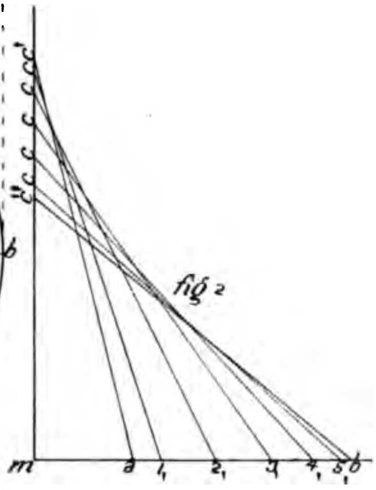


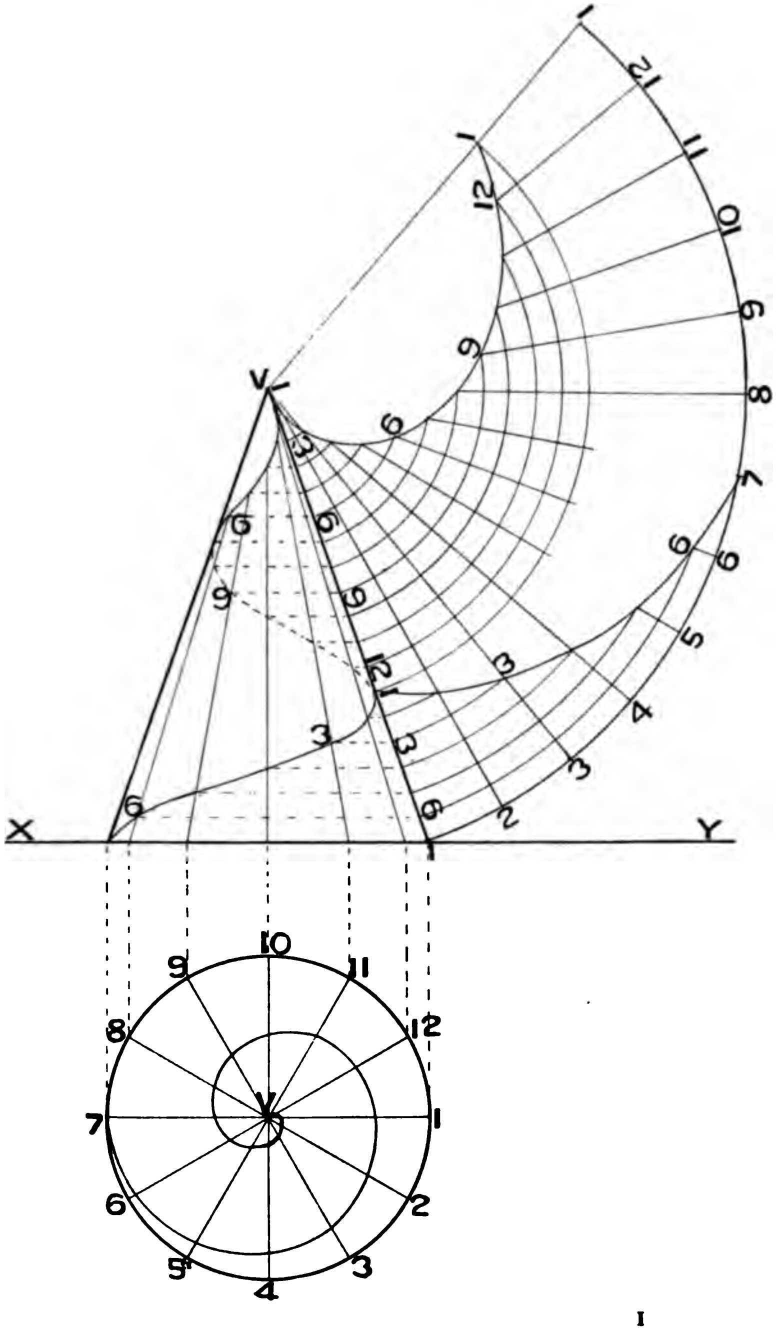
fig 2

No. 48, 1889. **A bead slides uniformly down a generator of a right circular cone (radius of base 1¼ in., height 3½ in.), starting from the vertex and reaching the base, while the generator traverses the conical surface one and a half times. Draw the developed locus of the bead.**

As the bead travels down uniformly from vertex to base without acceleration, the curve traced will be a helix. Divide the circular base of the cone into any number of equal parts, 12 in this case; divide, also, the slant side of cone into (12 + 6) parts, as the generator traverses the surface one and a half times; draw the generators in elevation, and project each of these divisions to its generator in turn, such as 6 to 6, 9 to 9, and 3 to 3, etc. The curve traced through these points in plan and elevation will be the locus of the bead. Its development will easily be found by developing the surface of cone and transferring each point on the generators to its corresponding generator on the development; thus, with V as centre, radius V1, describe circular arcs 1 2 . . . 1 equal to circumference of base of cone.* Draw the generators as shown, with V as centre, radii V1, V2, V3, etc., and draw arcs cutting generators V1, V2, V3, etc., respectively: the curve drawn through these points will be the developed locus. Note that the curve is in two portions; 1 6 9 1 is one complete turn, while 1 3 6 7 is the remaining half.

---

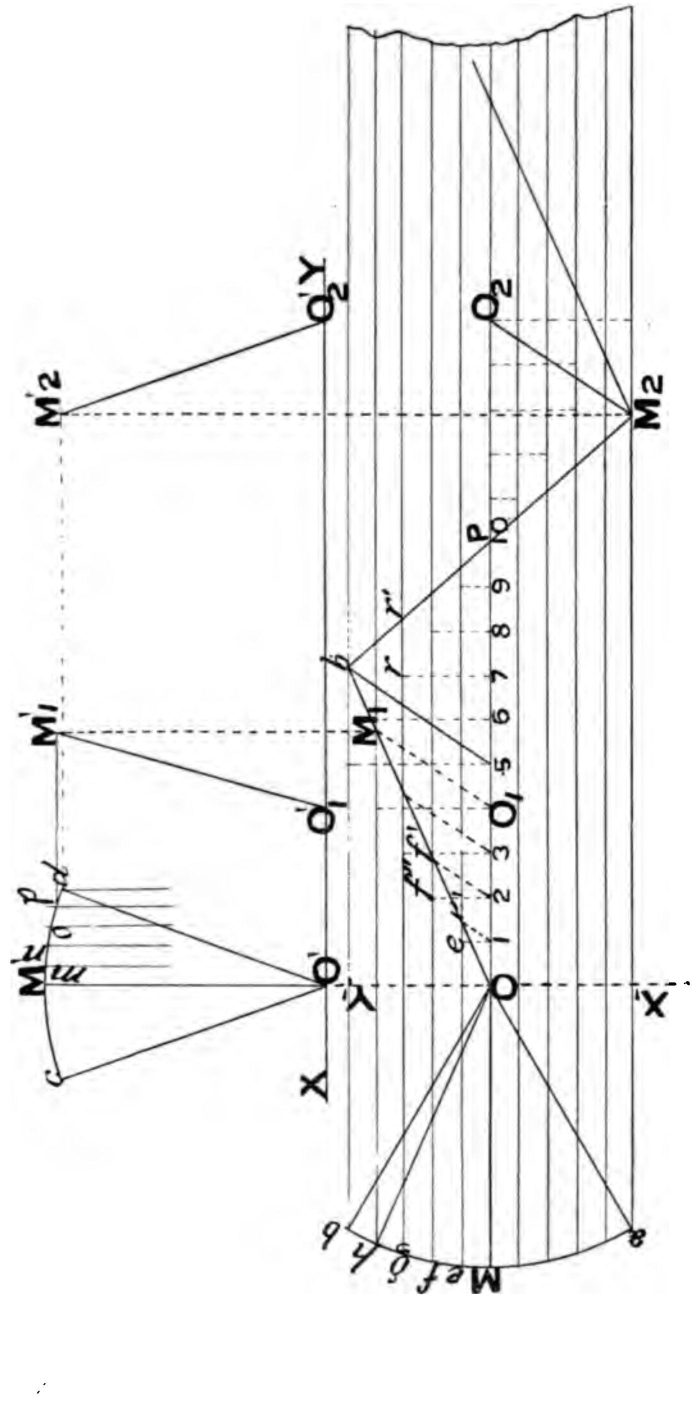
* Equal $2\pi r$, where $\pi = 3{\cdot}1416$, and r = radius.



I

No. 46, 1888. **The mainmast of a steamer running at a speed of 6 miles per hour, is 55 ft. high. The steamer is rolling 30° on each side of the vertical, and at the same time pitching 20° on each side of the vertical. The period of both roll and pitch is 10 seconds, and the motion takes place about the foot of the mast. Trace the plan of the locus of the top of the mast, starting from a vertical position (scale 1 in. = 20 ft.).**

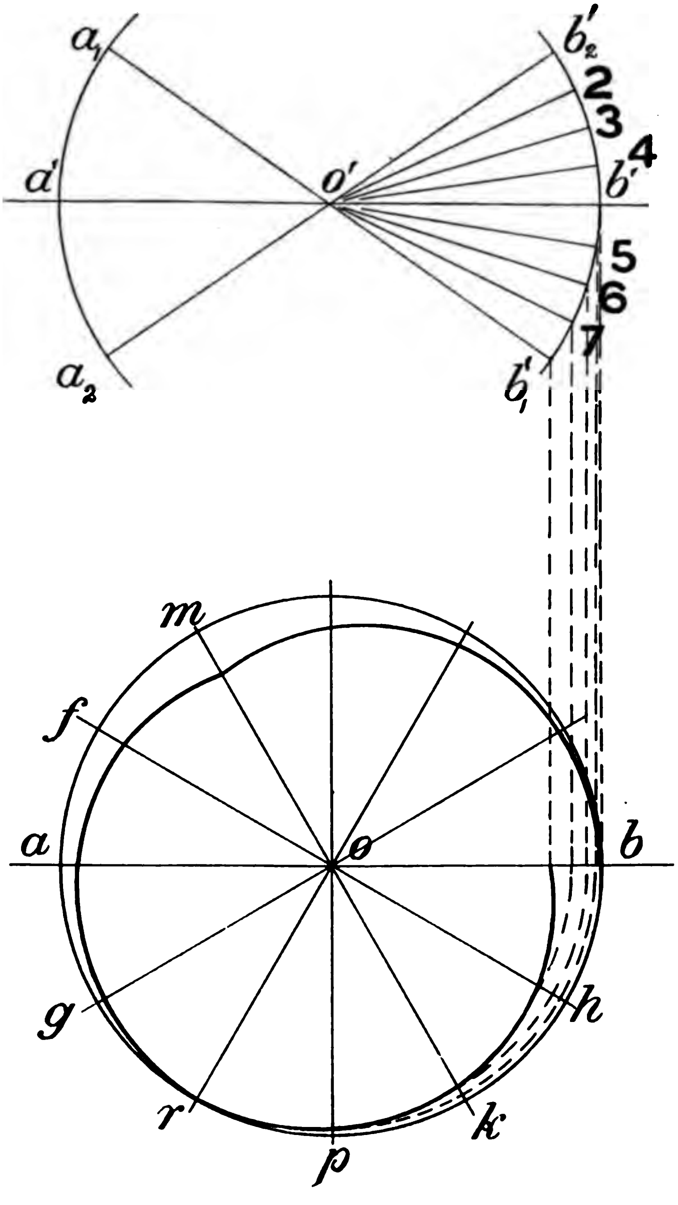
The path of the foot of the mast evidently will be a straight line, and as all motion is taking place about that foot, draw a line such as oo_1o_2, and set off lengths 1, 2, . . . 10, equal to the distance moved every second, viz. $\frac{6 \times 1760 \times 3}{60 \times 60} = 8\frac{4}{5}$ ft. per second, or 88 ft. in 10 seconds. Mark off OP = 88 ft. to the given scale, and divide OP into ten equal parts. The end of each of these divisions will therefore be the position of foot of mast after every second of time. Now on X_1Y_1 make an end view of the mast such as OM 55 ft. high; the arc *a*M*b* will be traced by the top of the mast when rolling through 30° on each side of the vertical, so that when the mast will have reached *b*, it will commence its backward journey to M and *a*. If this arc be divided into ten equal parts, each part will determine the position of the mast for each second during its motion. Make also a side view of the mast, as on XY, parallel to the direction of the motion; then on each side of *o*′M′ the mast will travel through an angle of 20°, describing the arc *c*M*d*. Divide this arc into ten equal parts as before, showing the amount the mast has pitched forward for every second. We may now commence to plot the path. After point M will have rolled through its angle in one second, and have moved forward one division, the plan of the top of mast will be at *e*, but the mast will also have pitched through its angle in one second, *i.e.* from M′ to *m*. Measure the perpendicular distance from *m* to *o*M, and plot it from *e* to *e*′; then *e*′ will be the plan of the top of mast at the end of the first second, due to rolling and pitching. Again, *o*M having rolled to O*f*, the plan of


M'2
O'2
Y
M'1
O'1
M'
u
m
p
o
d
c
O'
Y'
X
O2
M2
P
10
9
8
7
6
5
O1
3
2
1
b
r
r'
r''
M1
e
f
f'
f''
O
X'
a
b
h
g
f
e
M

f must be in a line drawn through f parallel to the direction of motion. At the end of 2 seconds it will be at f''. From f measure the distance $f''f'$, equal to the perpendicular distance of n' from $o'M'$, *i.e.* its pitch in 2 seconds; then f' is the plan of the top of the mast at the end of 2 seconds, due to rolling and pitching. Note that after the mast will have reached b, it will begin to pitch and roll backwards, until at P it is again vertical. Thus, when the foot of the mast is at point 7, say, after 7 seconds, the top of the mast will be at point r (ignoring the pitch); the pitching will have caused it to move forward a distance equal to the perpendicular distance of o from $O'M'_1$, for it will have returned through two divisions of its pitch. Measure, therefore, rr' equal to this distance; then r' is a point in the path. It will be seen that the plan of the locus is a straight line from o to b, and b to M_2. The backward path is very much shorter than the forward one, because the pitch takes place in the opposite direction to the forward motion of the ship. The plan and elevation of the mast is shown in two positions, as o_1M_1, o_2M_2. The student will have no difficulty in determining the other points in the path, if he takes into consideration the three motions every time, namely, forward, roll, and pitch, taking care to notice whether the pitch be backward or forward.

No. 47, 1887. **A rod ab is balanced at the point o, about which it also vibrates through the angle shown. While making two-thirds of a complete revolution, ab swings from the position $a_2b'_2$ to $a_1b'_1$. Draw the plan of the locus of one of the extremities of the rod during one whole revolution.**

Start from the position ab, a_2b_2. During the vibration from a_2b_2 to a_1b_1 in elevation, the rod will rotate through the angle *bhpam* in plan. Divide the arc b_2b_1 into eight equal parts, and the arc *bpam* into the same number. Draw the radii *oh*, *ok*, etc. Project the points b'_2, 2, 3, etc., to the line ab, and with o as centre describe circles cutting the above radii. The curve joining the points of intersection of radii and circles is the plan of the locus of the right-hand end of the rod.


a₁
b'₂
2
3
4
a'
o'
b'
5
6
7
a₂
b'₁
m
f
a
o
b
g
h
r
k
p

No. 58, 1894. **A rod *ab* rotates round a fixed vertical axis *oo′*; the angular velocity diminishes uniformly, and becomes zero after two revolutions; the angle θ, which the rod makes with the axis, uniformly diminishes, and also vanishes after two revolutions. Trace in plan and elevation the locus of the point *a*.**

As the angle θ diminishes uniformly, the angular velocity of OA in the plane of the paper will be uniform; hence the arc *a′m* may be divided into any number of equal parts—in this case twenty-four—each part of which will require $\frac{1}{24}$ of the time of motion for the point A to pass over it.

The angular velocity about the axis *o′m* diminishes uniformly to zero; hence the angular distance moved through by the arm and axis will be given by the expression—

$$\phi = Vt + \tfrac{1}{2}at^2$$

where ϕ is the angle turned through, in time t, the initial angular velocity being V, and a the angular acceleration. For the sake of convenience, it will be advisable to work backwards from the finish to the start, and then the above equation becomes—

$$\phi = \tfrac{1}{2}at^2$$

We will select the unit of time as $\frac{1}{24}$ of the whole time, and the unit of angular measure the degree. At the end of two complete revolutions, $\phi = 720°$, and $t = 24$; then the above equation gives—

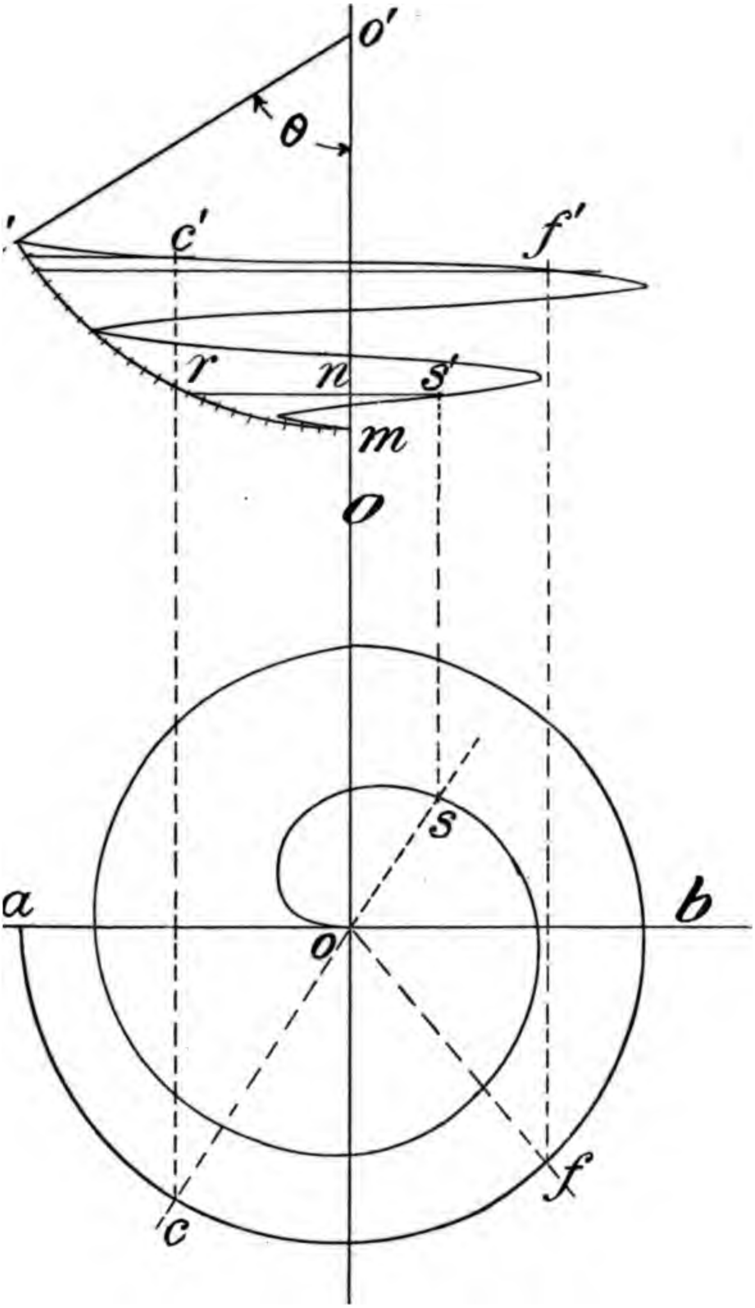
$$720 = \tfrac{1}{2}a \times 24^2$$

and $a = 2{\cdot}5°$ per interval during each interval; and the above equation becomes—

$$\phi = 1{\cdot}25t^2$$

Putting in the several values of t, *viz.* 1, 2, 3 . . . 23, we obtain corresponding values for ϕ.

Plotting these values we get the locus of the end A of the arm; thus at the end of the tenth interval $\phi = 125°$. Set off the $\angle\, aos = 125°$; the point required will lie in *os*.

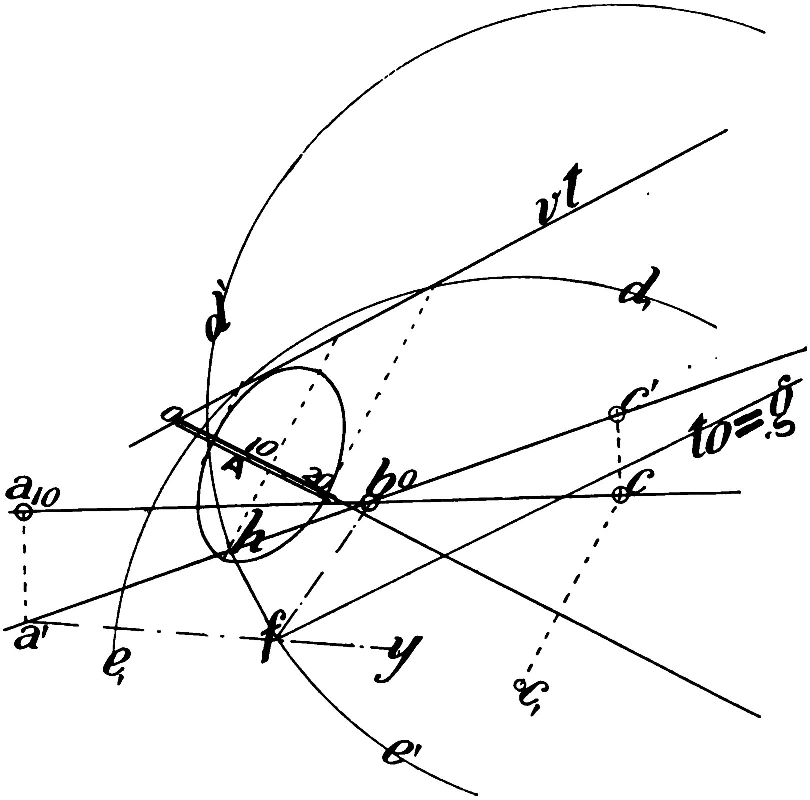

o'
θ
'
c'
f'
r
n
s'
m
O
s
a
b
o
f
c

The end of the tenth interval from *m* is at *r*, and the horizontal distance between *r* and the axis is *rn*; hence set off *rn* along *os*, and the point *s* is found. The elevation of *s* is immediately above *s*, and in the same horizontal line as *r*; hence it is at *s'*. In the same way, when $t = 22$, $\phi = 610°$. Measuring off 610° from *oa* in watch-hand direction, we obtain *of*. The length *of* equals the horizontal distance of the end of the twenty-second interval from the axis *o'm*. The elevation of *f* is at *f'*. After plotting the plans and elevations of the several positions of A, the two curves are drawn as shown in the figure.

No. 55, 1891. **A plane and two external points, A, B, are given. Determine the locus of a point moving in the given plane so that its distances from the two given points are in the constant ratio 2 : 3 (unit 0·1 in.).**

The points a_{10}, b_0 and the plane marked A are given. The locus required will be the section made by the given plane, of the sphere, which is the locus of all triangles on the base *ab*, and the ratio of its sides, 2 : 3.

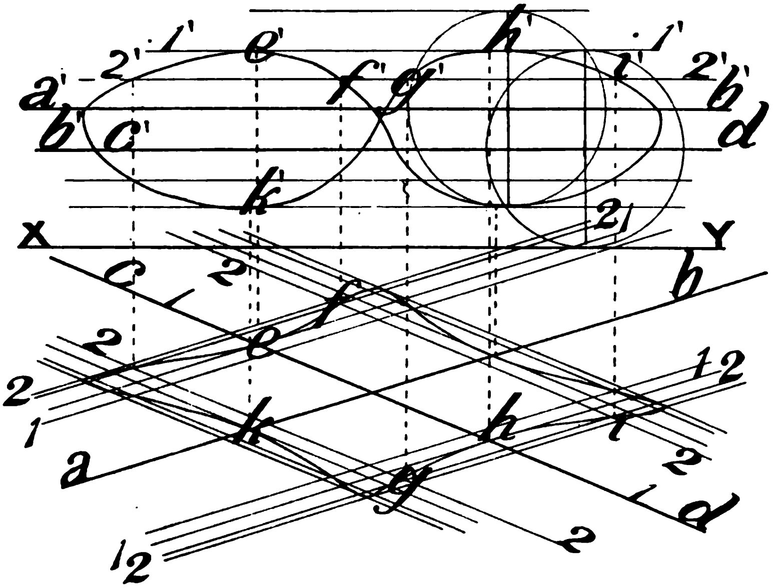
Find *a'b*, the elevation of *ab*, and on it construct any triangle, sides $af = 3$, $bf = 2$; bisect the exterior angle *bfy*, and produce to meet line *a'b* in point *g*; draw *fh* perpendicular to *fg*, meeting *a'b* in point *h*. Then on *hg* describe the circle *d'e'* with centre *c'*; its plan will be *c*. Now imagine this circle, which is the locus of the vertex of all triangles on *ab* in its plane having sides 2 : 3, to revolve about its diameter *hg*, thus generating a sphere; then every point on this sphere will be distant from *a* and *b* in the ratio of 2 : 3. Now determine the elevation of sphere d_1e_1 and plane, using the scale of slope as *xy*. The centre of sphere will be c_1 below the *xy*. The plane whose vertical trace is *ovt* will cut the sphere in a circle whose plan is the ellipse shown. This ellipse is the plan of the locus on the given plane of all points distant 2 : 3 from A and B respectively.


vt
d'
d,
a
10
A
30
b0
c'
c
to=g
a10
h
a'
e,
f
y
c,
e'

No. 46, 1887. **Trace the locus of the centre of a circle 2 in. in diameter, touching the two given lines** *ab*, *a'b'*; *cd*, *c'd'* **supposed indefinitely produced.**

The lines *ab*, *a'b'*, *cd*, *c'd'*, and XY are given. It will be noticed that the lines are horizontal. The locus of the centre of a circle 2 in. diameter touching both lines, evidently will be the intersection of two cylinders having the given lines for axes and 2 in. in diameter, for then every point on the curve of intersection must be 1 in. away from each line or axis.

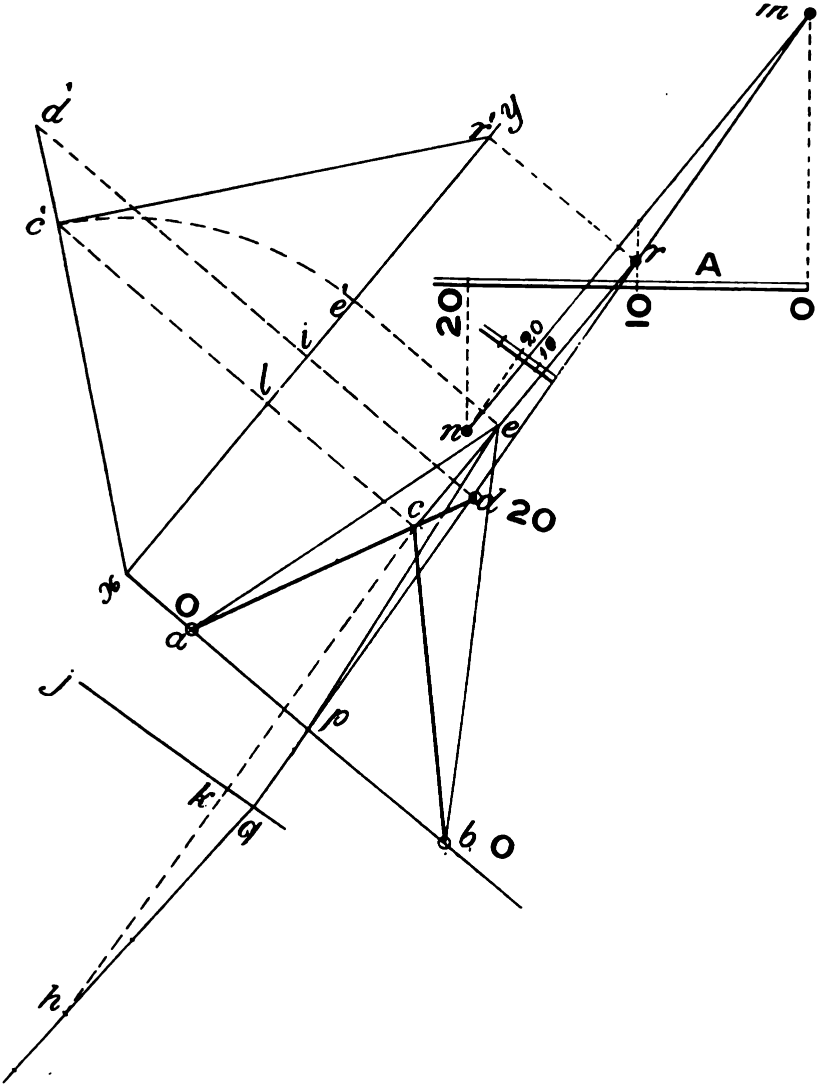
Commence by drawing the projections of the cylinders, and take a series of horizontal cutting planes to cut both cylinders in generators parallel to their axes; their crossing-points in plan are points on the curve. Thus take a vertical section through the cylinders perpendicular to axes, rotate it into the V.P., as shown by the two circles in elevation; and let the first section be represented by line 1 1 tangential to the lower cylinder axis *cd*, cutting the upper cylinder whose axis is *ab* in the two generators 1 1 (seen in plan). Their crossing-points *e*, *h*, with the upper generator of cylinder axis *cd*, are two points on the curve. Similarly, the plane 2, 2 cuts both cylinders in two of their generators, as 2 2, 2, 2; their crossing-points 2, *f*, *i*, *g* are four points on the curve, and so on for any number of points. The freehand curve drawn through these points will be the locus of the centre of the circle as required. The elevation is easily obtained, for every point will have the elevation on their respective cutting planes; as *eh* on line 1 1 to *e'h'* on 1'1', 2*fgh* on lines 2 2 will be on cutting plane 2'2' at 2'*f'g'h'*, etc. The curve drawn through these will be its elevation.


2'
1'
e'
h'
1'
i'
2'
a'
b'
f'
g'
b'
c'
d'
k'
2
1
X
Y
c
1
2
f
b
2
e
2
1
12
k
h
i
a
g
2
1
d
2
1
2

No. 58, 1893. **The lines acd, bc meet at c. Determine the locus of all points in plane A equidistant from the two given lines (unit 0·1 in.).**

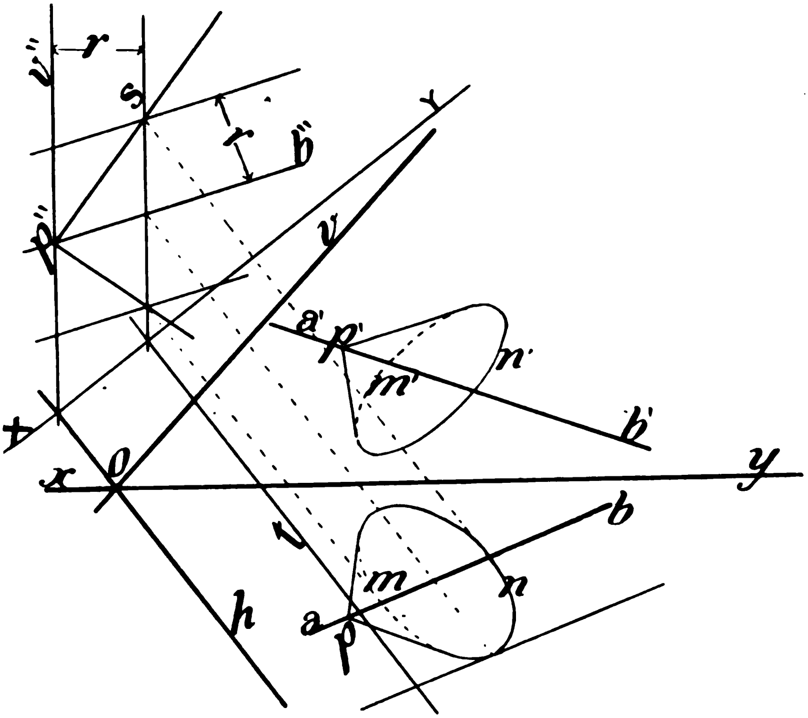
The heights of the points a, b, and d are given as 0, 0, and 20. If a plane is made to contain c, and be perpendicular to the plane of acb, and at the same time bisect the angle between ac and bc, this plane will contain all points which are equidistant from a and b. The intersection of this plane with the plane whose scale of slope is given, will contain all points in the given plane which are equidistant from a and b.

Construction.—Join ab; this is the horizontal trace of the plane containing the two lines ac and bc. Take a new ground-line xy perpendicular to ab. Project up $d20$ perpendicular to xy, and set up id' equal to 20 units. Join $d'x$, and get c', the elevation of c. Rotate the triangle abc about ab down to the ground, as shown at aeb. The bisector of the angle aeb is ep. As the triangle is rotated back to its original position, ep will come into the position cp, the point p remaining stationary; therefore p will be a point in the horizontal trace of the plane which bisects the real angle acb. To get another point in the horizontal trace—a line through c and perpendicular to the plane of abc will lie in the bisecting plane; hence through c' draw $c'r'$ perpendicular to xd'. The plan of r' will lie in a line through c perpendicular to the horizontal trace ab, because if a line is perpendicular to a plane, the projections of the line will be perpendicular to the traces of the plane. Join pr; this is the horizontal trace of the bisecting plane. Take a new ground-line qj perpendicular to pr, and project ckh perpendicular to it, setting up $kh = lc'$. Join qh; this is the end elevation of the bisecting plane, from which the scale of slope is obtained. Now get the line of intersection of this plane with the given plane. Draw horizontals at levels 20 in each plane intersect in n; and horizontals in each plane through o intersect in m: therefore nm is the plan of the line of intersection, and therefore the plan of the locus of points in the given plane equidistant from a and b.


m
d'
x'y
c'
r
A
20
10
0
e'
i
20
10
l
n
e
c
d
20
x
O
a
j
p
k
q
b
O
h

No. 56, 1890. **A line and plane are given. Determine the projections of the surface locus of a point moving so that it is always equidistant from the given line and plane.**

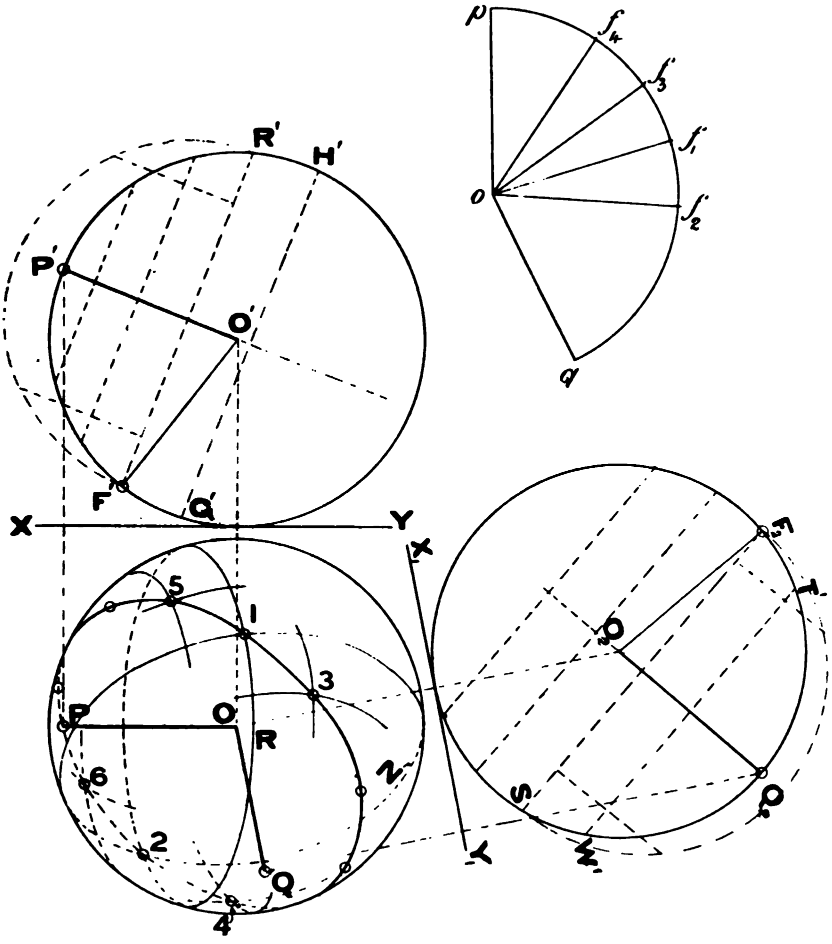
The plane *voh*, line *ab*, *a'b'*, and the *xy* are given. The surface locus must evidently be a cone with its vertex on the intersection of the line *ab* with the given plane, as at *pp'*. Commence, therefore, by taking XY perpendicular to the H.T. of the given plane, and determine its V.T. as shown at *v''*, and also the elevation *a''b''* of the given line *ab*. Now consider the line *a''b''* as the axis of a cylinder of any radius *r*. Take another plane parallel to the first one, and of distance away equal to the radius of cylinder *r*, and the section made through the cylinder by this plane will be the ellipse *mnm'n'*, and will at all points be equidistant from the line and plane. Now tangents drawn from point *p''*, the intersection of line and plane to this elliptic base, will form a conical surface extending both ways from point *p''* indefinitely; determine its plan and elevation *m'n'*, *mn*, relatively to the given *xy*, which is the required locus.


v''
r
s
r
b''
p''
v
t
a'
p'
n'
m'
b'
x
o
y
b
m
n
h
a
p

No. 59, 1891. **The plan of a sphere is given (centre O), and also the plan of two points, P and Q, on the upper surface of the sphere. Determine the plan of the locus of a point F, moving on the surface of the sphere so that the sum of the angles POF, FOQ is constant and equal to 154°.**

The plan is given at the left-hand lower corner of the figure. Take a ground-line XY parallel to PO, and find the elevation P′ of the point P. Take another ground-line X_1Y_1 parallel to OQ, and obtain the elevation Q_2 of the point Q. Draw the angle $poq = 154°$ (upper right-hand corner of figure), making $po = oq = O'P'$. Take any point f_1 on the arc pq, and set off the angle $P'O'F'$ = the angle pof_1; then all possible positions of F′ (when $P'O'F' = pof_1$) will be traced out by F′ as P′O′F′ is rotated about P′O′ as axis. The locus of F′ will then be a circle whose elevation is F′R′, and in plan the ellipse R12. Set off the angle $Q_2O_2F_2$ equal to the angle qof_1; then the locus of F_2, as it is rotated about O_2Q_2 as axis, is F_2S in elevation and the ellipse Z1 2 in plan. The intersections of these two ellipses are two points on the locus of F on the surface of the sphere. These points of intersection are 1 and 2.

Next take any other point f_2 on the arc pq, and set off $P'O'G' = pof_2$. Similarly, make $Q_2O_2T' = qof_2$. Then the locus of T′ and the locus of G′ intersect in plan at the points 3 and 4. Take a further point f_3 on the arc pq, and the intersection of the loci will be at 5 and 6. After finding a few more intersections, the points may be joined up by a freehand curve, and the final locus of F, the ellipse 5 1 3 4 2 6 will be obtained. It will be noticed that, after drawing the first pair of ellipses, it is not necessary to complete any of the remaining ellipses, but only those parts near an intersection.


p
f4'
f3'
f1'
o
f2'
q
R'
H'
P'
O'
F'
Q'
X
Y
x
5
1
3
O
P
R
6
N
2
Q
4
F2
T
O
S
W
Q
y

No. 53, 1895. **A circle of 27 units diameter lies in a plane whose horizontal trace is CD. The circle touches the horizontal plane, and has its centre in plan at *a*. P_{15} is a given point. Find a chord of the circle such that the triangle formed by joining P with the extremities of the chord will be equilateral. How many solutions are there? (Unit = 0·1 in.)**

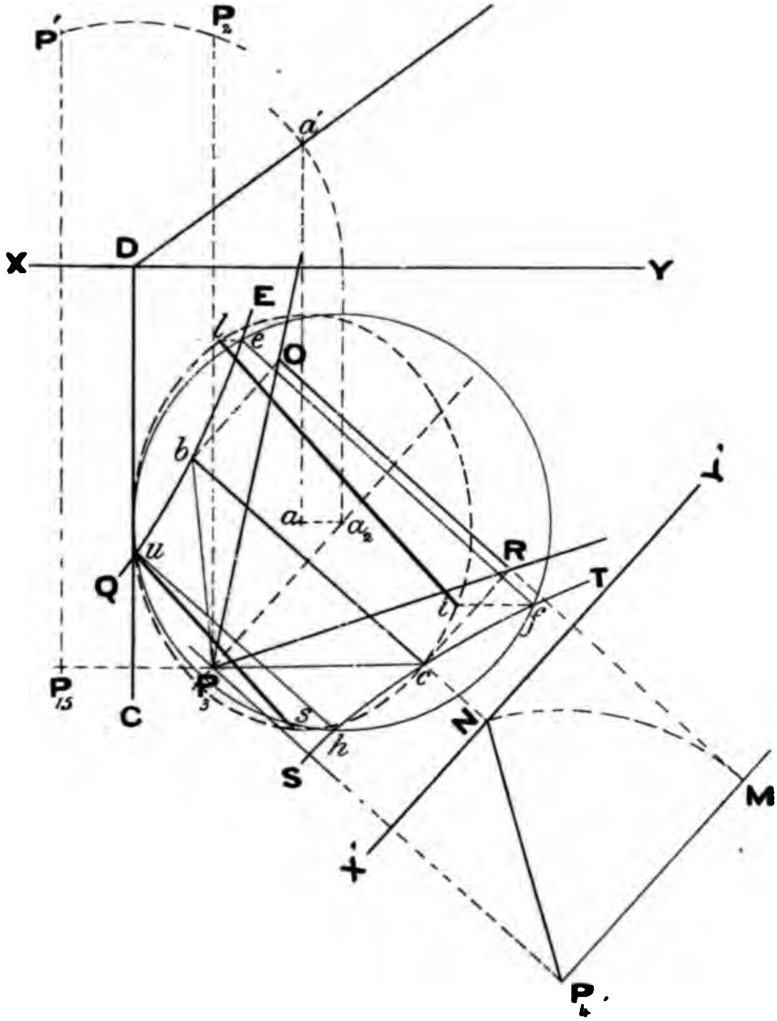
Take a ground-line XY perpendicular to DC. Through *a* draw a vertical, and with D as centre and $\frac{27}{2}$ units radius describe an arc cutting the vertical in a'. Join Da'. This is the end view of the plane containing the circle. In the diagram given to candidates at the examination, the point *a* was wrongly placed in such a position that a' came in XY. The examiners mention this in their report, and the point *a* in the accompanying diagram has been moved slightly to show the meaning of the question.

Get the elevation P' of P. Now rotate the plane down to the ground, and rotate the point P' through the same angle about D to P_2. Describe the circle with a_2 as centre, touching CD. The plan of P_2 is P_3. Join P_3a_2.

Now, P_3 is the plan of the vertex of an equilateral triangle, and, from the nature of the problem, P_3a_2 must bisect the plans of all the vertical angles of equilateral triangles whose bases are chords of the given circle. Find the loci of the ends of the bases of all possible equilateral triangles. Thus take a new ground-line X'Y' parallel to P_3a_2, and get P_4, the elevation of P_3. Draw P_4M parallel to X'Y'. At P_3 make each of the angles a_2P_3O and a_2P_3R equal to 30°. The lines P_3O and P_3R will be two sides of a horizontal equilateral triangle taken at random, of which OR is the base.

Swing this triangle down about an axis P_3P_4 until the base RO comes into the ground at *cb*; then cP_3b is the plan of *an* equilateral triangle which fulfils the conditions above, except that the base does not terminate in the circumference of the circle.

In the same way find the bases of other triangles. It will be found that their extremities lie on the curves ST and QE. In other words, if the lines P_3O and P_3R (which include an angle of 60°) be rotated about a horizontal axis whose plan is P_3P_4, the locus of their traces will be the two curves. These curves cut the circle in *f*, *e*, *u*, and *h*; therefore *ef* and *uh* are the chords required, and there are two solutions. Rotate the circle back into its original position (centre *a*), and the chords move to the positions *li* and *us*.



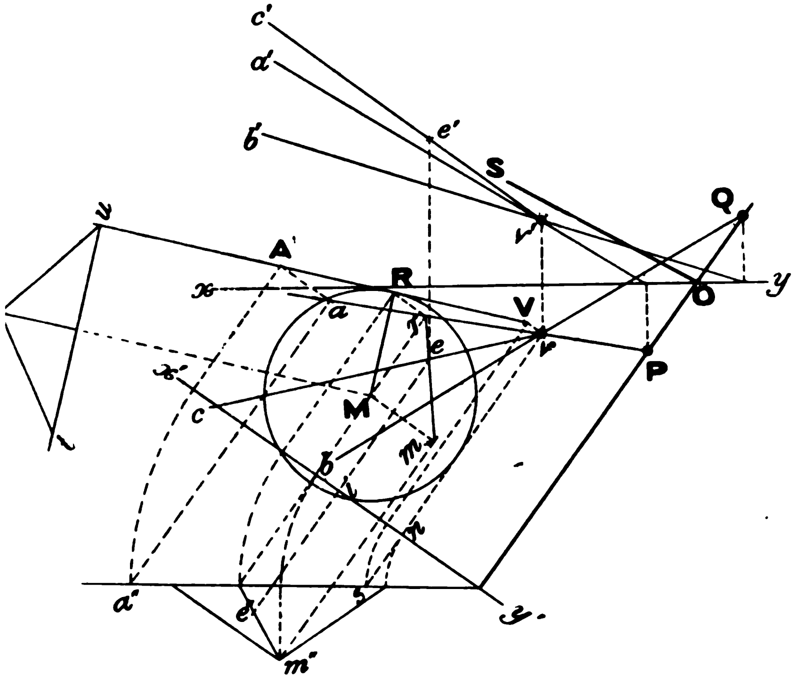
No. 55, 1892.

CHAPTER IV.

SOLID GEOMETRY—LINES AND PLANES.

Three lines, AV, BV, CV, intersecting at V, form a trihedral angle. The projections of two of them (AV, BV) and the plan of the third line (VC) are given. Assuming that the dihedral angle at AV is 35°, determine the elevation of CV.

The projections of the given lines are *av*, *a′v′* and *bv*, *b′v′*, together with the plan *cv*. It is required to find the elevation *c′v′*. Or the problem may be stated thus: given the plane containing AV and BV, put another plane inclined to the former at an angle of 35° and intersecting it in AV. First obtain the traces of the plane containing AV and BV. They are PO and SO. Rotate this plane about PO down to the ground by means of the end elevation on *x′y′* as ground-line. The line *va* becomes VA on the ground. Now place a cone with base resting on the ground and touching AV at any point R. Let the angle at the base of cone be 35°. The plan of cone is shown in the figure with M as vertex. This is obtained by the auxiliary elevation on *tu* as ground-line perpendicular to VA. The angle *tum′* equals 35°. The line MR is a generator of the cone surface, and is perpendicular to AV; it therefore lies in a plane containing AV, and inclined at 35° to the ground. Rotate the plane whose horizontal trace is PO back to its original position, together with the cone. MR comes back to *mr*. Now, *mr* is the plan of a line lying in a plane inclined at 35° to the plane containing AV and BV; also *cv* is in the same plane, therefore *cv* and *mr* must intersect. Let *e* be the plan of the intersection. Its height above the ground is obtained from the auxiliary elevation, and is *e″i*. Set up this height above *xy*, and we get *e′*. Join *e′v′*; this is the elevation of CV.


c'
d'
b'
e'
S
Q
u
A'
R
x
y
O
V
v'
v
a
e
P
x'
c
M
m
b
n
a''
e''
m''
y'

No. 54, 1892. **The two given lines AR, CE intersect. Assuming this intersection to be inaccessible, draw a line through the given point P which shall pass through it.**

Let AR and CE be two lines intersecting at some point O, which is off the paper. Let P be any point in space. Cut the lines by any plane containing the point P; thus, take any two points A and C on the two given lines, and join them in plan and elevation; also join the points A and C to P: then the above plane containing P cuts the given lines in A and C. Cut the given lines by another plane parallel to the ACP: thus, take any point r on ao, and project upwards to get the elevation r'; through r draw re parallel to ac, and through r' draw $r'e'$ parallel to $a'c'$; similarly, draw ei and ri parallel to cp and ap; and, finally, draw $r'i'$ and $e'i'$ parallel to $a'p'$ and $c'p'$ respectively. Join $p'i'$ and pi; these are the projections of a line through P which will pass through the intersection of the other two. For the plane REI is parallel to the plane ACP, and parallel planes cut all lines radiating from the same point in the same ratio. Now, because re is parallel to ac, and ce and ar intersect in O, therefore—

$$\frac{ac}{re}=\frac{ao}{ro}$$

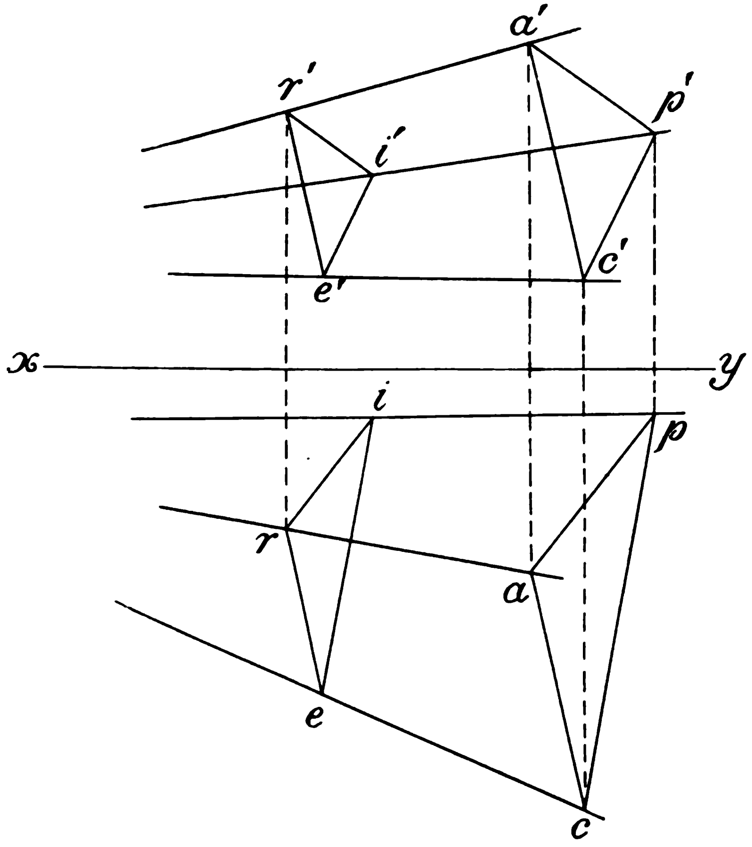
Also because ri was drawn parallel to ap, and ei to cp, then—

$$\frac{ac}{er}=\frac{ap}{ri}$$

Therefore, from above—

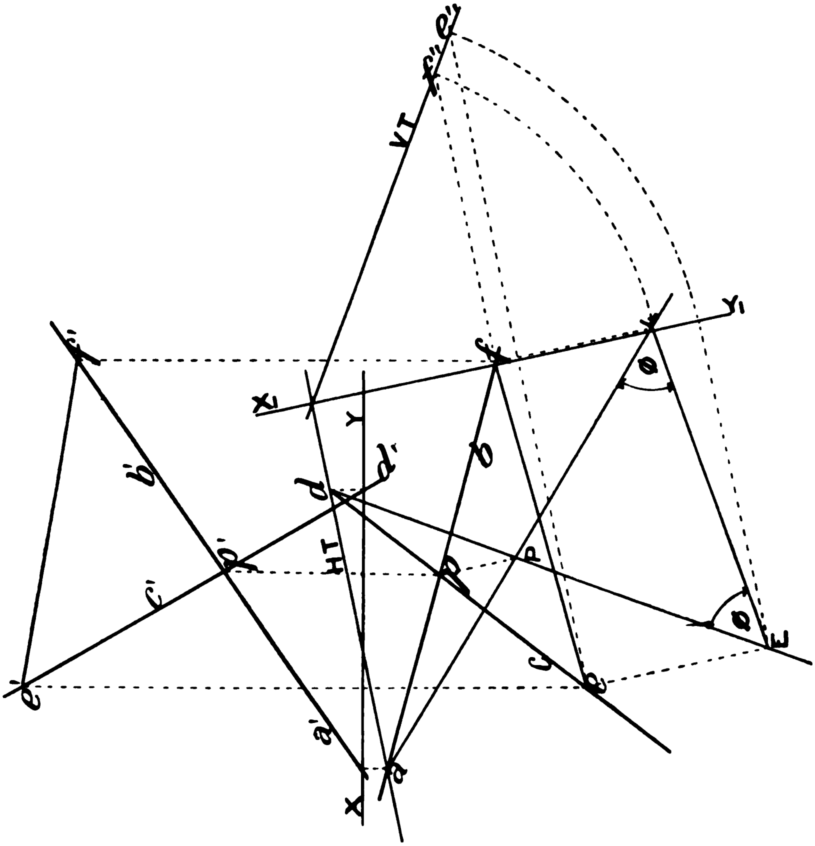
$$\frac{ap}{ri}=\frac{ao}{ro}$$

and hence ar and pi intersect in O, that is, the line pi, and consequently PI, passes through the point of intersection of the given lines CE and AR.


a′
r′
p′
i′
c′
e′
x
y
i
p
r
a
e
c

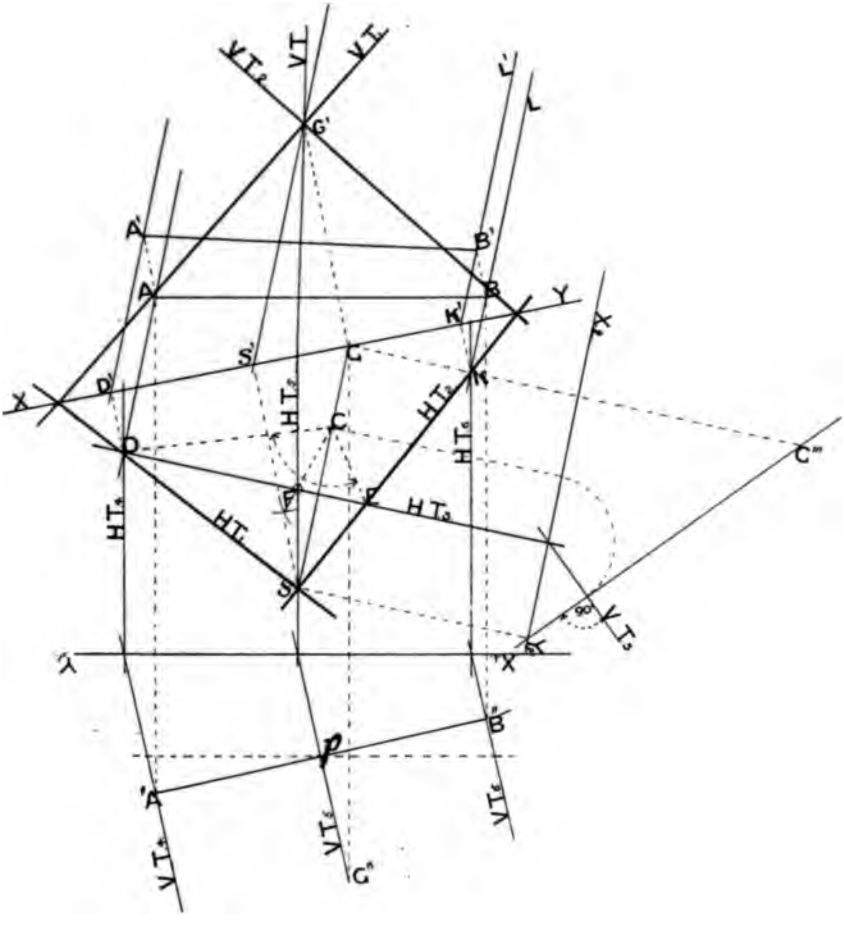
No. 54, 1890. **Two intersecting lines are given. Determine the projections of a line $4\frac{1}{4}$ in. long, with its extremities on these lines respectively, and inclined at the same angle to both.**

Let *ab*, *a'b'*, *cd*, *c'd'* be the given lines. The line required must lie in the plane containing the given lines; determine, therefore, in the usual way, a plane whose traces are HT and VT to contain the given lines. Now make a new elevation of the plane and lines on the ground-line X_1Y_1, taken at right angles to HT, as shown at VT, *f''*, *e''*. As *pp'* is the intersection of the given lines, the height of *p'* above X_1Y_1 will be equal to its height above the original XY (*p''* is not shown in the drawing). Rotate this plane about its H.T. to the H.P., taking the lines with it to *a*P, *d*P. Produce the lines *a*P, *d*P indefinitely, and place a line EF $4\frac{1}{4}$ in. long to meet them in E and F, and making equal angles with each, *i.e.* $\angle$ EFP = $\angle$ FEP. Now rotate the points EF back to their proper position on the given lines, as at *ef*, *e'f'*; then *ef e'f'* is the required line. Another line, however, may be placed on the lower side of *p*, to satisfy the given condition; but it would come below the H.P.


e''
f''
VT
Y₁
X₁
Y
f'
f
φ
b'
b
d'
d
HT
g'
g
P
c'
c
e'
e
E
φ
a'
a
X

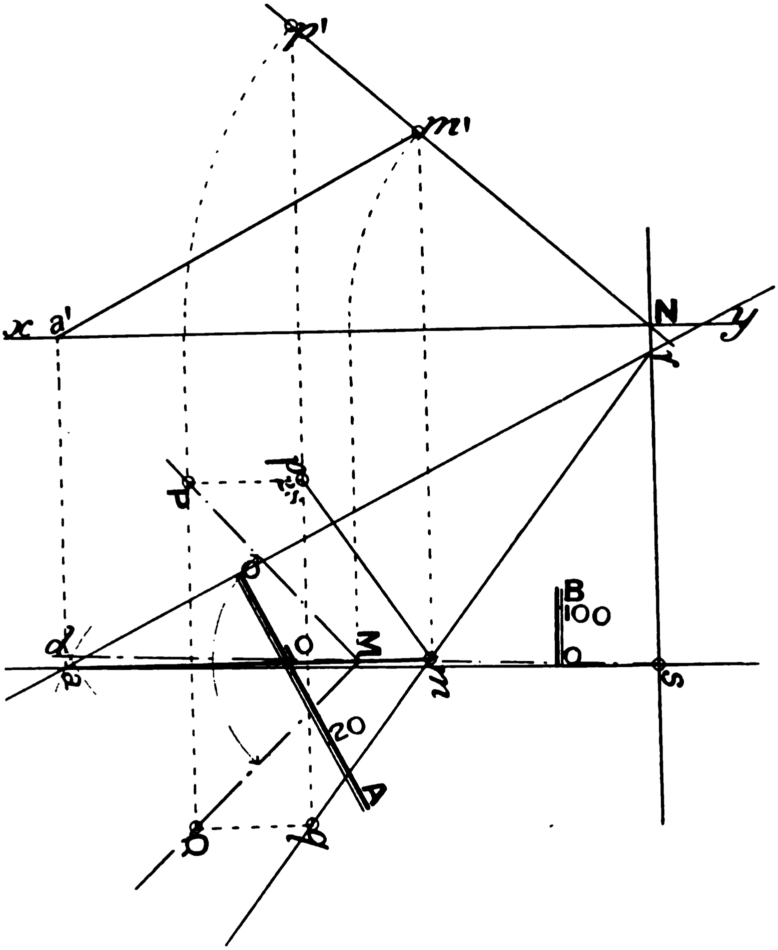
No. 45, 1887. **Determine the projections of a line $2\frac{1}{2}$ in. long, with one extremity in each of the given planes, and inclined at the same angle to both. The centre of the line to be $\frac{3}{4}$ in. above the horizontal plane.**

The planes whose traces VT_1, HT_1, VT_2, HT_2 are given. The line required is to be equally inclined to both planes; therefore it must be perpendicular to the plane bisecting the dihedral angle, and its extremities must be in the intersections of the other two planes with the given planes, and at a distance of $1\frac{1}{4}$ in. away from the plane bisecting the dihedral angle on both sides. Commence by determining the dihedral angle of the given planes, and bisect it with the lines CF; then F is a point in the HT of the plane of bisection. S must be another point. Draw HT_5 through SF, and on X_1Y_1 perpendicular to HT_5 determine VT_5, and parallel to it and $1\frac{1}{4}$ in. away draw VT_4 and VT_6, HT_4, HT_6; find their intersection with the given planes, as DA, D′A′, KL, K′L′. Now find point p $\frac{3}{4}$ in. above X_1Y_1, and through it draw line A″B″ perpendicular to VT_5; determine the projections of these points on the intersections as at AB, A′B. Then A′B′AB is the line required.


VT2
VT
VT1
L'
L
G'
A'
B'
A
B
Y
X
K'
K
S'
C'
D'
X
C
HT2
HT6
O
F
E
HT3
HT4
HT1
S
C'''
90°
VT5
S1
Y
X
B''
P
A''
VT4
VT5
VT6
G''

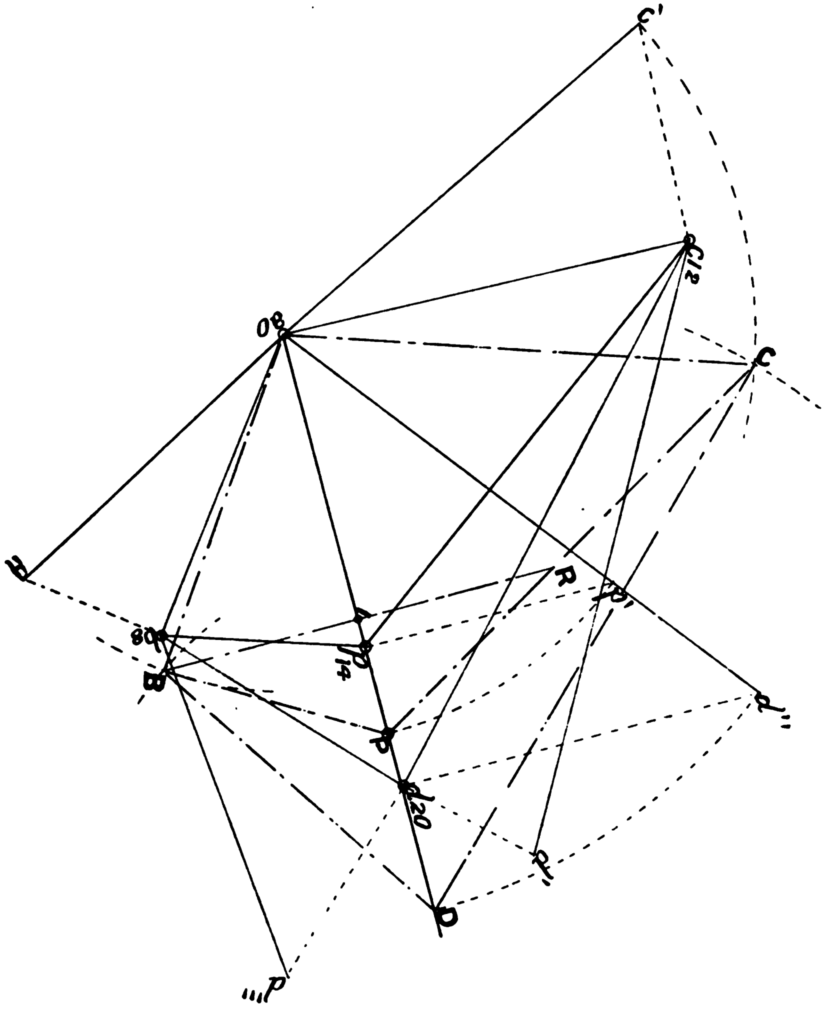
[illegible] **A plane is given by its scale of slope marked A;** *am* **is the plan of a line in the plane, and P is an external point. Draw through AM a plane making equal angles with the given plane and with the line PM$_1$. Unit 0·1 in. (N.B.—M is a definite point on the line AM$_1$.)**

It is well to notice that the plane required must bisect the angle between the line PM and the given plane, and not the angle between a plane containing PM, AM, and the given plane. We must therefore find a line in the given plane which bears the same relation to line *am* as *pm* does. This is easily found by passing a plane through *pm*, having its horizontal trace perpendicular to *am*. The intersection *mq* of this plane with the given plane will satisfy that condition, because the lines *mq* and *mp* are contained by the same plane whose horizontal trace is perpendicular to *am*. To do this, make an elevation of line *am*; (the index of *m* may be obtained from the scale of slope A) and point *p*, on a ground-line parallel to *am*. Draw the vertical trace through *p'm'*, and produce it to meet *xy* in N; N*s* drawn perpendicular to *xy* will be its horizontal trace, cutting the horizontal trace *ar* of the given plane in point *r*; then *rmq* will be their intersection. Next *rabat* this plane to the horizontal plane, taking the line *mp* and the intersection *mq* with it down to MPQ. Now bisect ∠ PMQ with the line *d*M, and produce it to meet N*s* in *s*; then *s* will be a point in the H.T. of the required plane. *a* is also another point. Join *as*; then *as* is the horizontal trace of the plane shown by its scale of slope B, which makes equal angles with PM and the given plane A.


p'
m'
x
a'
N
y
r
p
p₁
o
o
M
B
100
0
a'
a
m
s
20
A
o
q

No. 44, 1889. **ABCD are four given points. Find the point in AD at which the segments AB, AC subtend equal angles (unit 0·1 in.).**

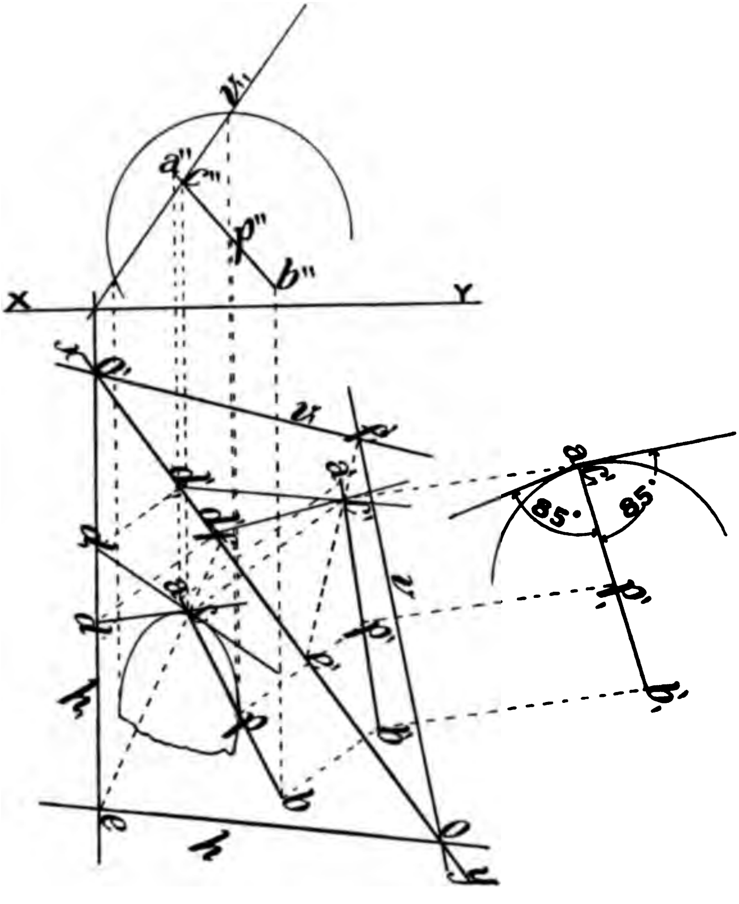
The points are given by their indexed plans *a*0, *b*8, *c*12, *d*20. In all problems of this class it is invariably found expedient to *rabat* all the points to the horizontal plane, then the true shape of the figure is obtained, as *a*BCD. Thus determine the true length of *ad*, as *ad''*, and rabat to the horizontal plane at D; find true length of *ac* as *ac'*, and of *cd* as *cd'*, *ab* as *ab'*, *bd* as *bd'''*, *i.e.* working to the difference of level of points, thus height of *d'* is taken as 20 − 12 = 8, etc. With these true lengths draw triangles *a*CD, *a*BD, which are the true shapes lying on the H.P.; then draw B1 perpendicular to *a*D and produce it to R, making 1R = 1B; next join R to C, and produce it to cut *a*D in point P: then ∠ CP*a* = ∠ BP*a*, which are the angles subtended by the segments. Next rabat the point P back to its proper height as at *p'* and its plan *p*; then *p* is the required point, index 14.


c'
c12
o8
C
a
8b
B
p14
R
p'
P
d''
d20
d'
D
p'''

No. 55, 1890. **Two lines, AB, CD, include an angle of 85°; AB lies in the given plane *voh*, and *a'b'* is its elevation; CD lies in the given plane *v'o'h'*. Determine the projection of CD.**

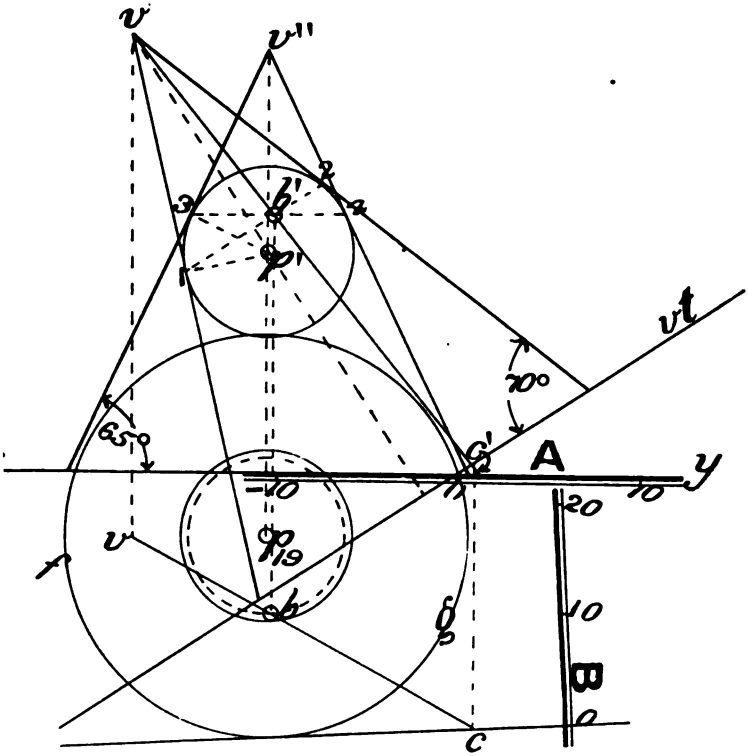
The planes labelled *v'o'h'*, *voh*, line *a'b'*, and the ground-line *xy* are given. Commence by finding the intersection *ef*, *e'f'* of the two given planes. The line *a'b'* cuts the intersection in point *c'*; determine its plan, *c*. The required line CD must start at the point *c*, for point *c* is in both planes, and it must also be the intersection of the plane *v'o'h'* with a cone axis *cb*, *c'b'*, semi-vertical angle of 85°; this will readily be found by finding true length of *cb*, as $c_1'b_1'$. Make a cone semi-vertical angle of 85° as shown, and in it inscribe a sphere centre p_1'; then the plan *v'o'h'* will cut this sphere in a circle, and the tangent drawn to this circle from the vertex c_1 will be the line required. Make, therefore, an elevation of the plane, line, and sphere, taking XY perpendicular to *o'h'*, as shown, the plane cutting the sphere in a circle; its plan is shown by the part ellipse. Now draw *cd*, cd_1, tangents to this ellipse, and find their elevations: then *cd*, *c'd'*, $cd_1c'd_1'$ will be two lines fulfilling the above condition.

The student may notice that the vertical angle of the cone is 170°, making its outline very nearly a straight line, and it is, therefore, very difficult to get the inscribed sphere. We would, however, recommend the student to alter the data so that he may understand the principle better. Let the lines CD, AB to contain an angle of about 47°, when the point *c* will not be so near the circle in section, and he will more readily be able to draw the tangents.


v'
a''
c''
p''
b''
X
Y
x
a'
v
p'
a
d
h
e
p
b
o
y
85°
85°
c'
b'

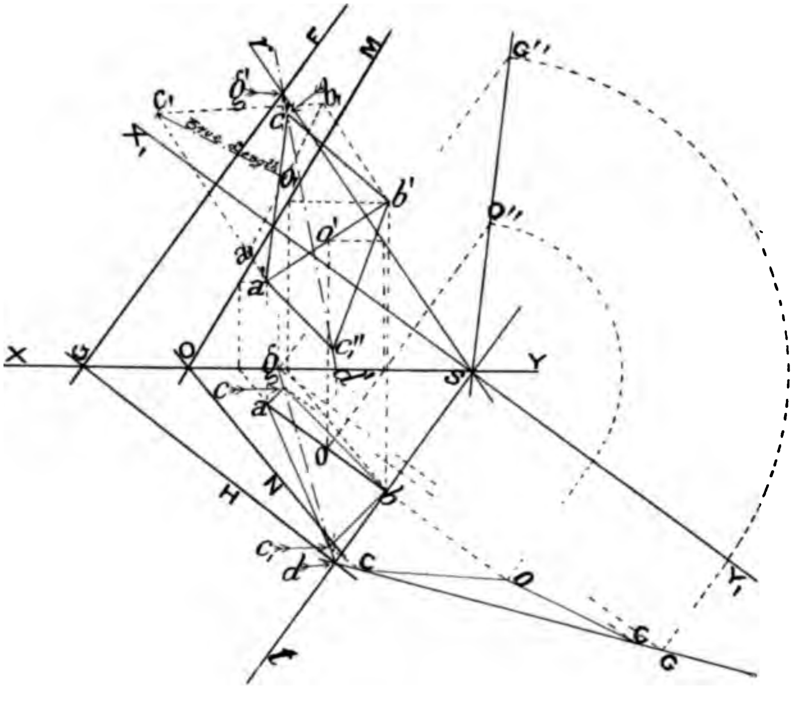
No. 45, 1889. **Determine a plane inclined at 65°, making 70° with the given plane, and $\frac{3}{4}$ in. distant from the given point P (unit 0·1 in.).**

The scale of slope marked A and point *p*19 are given. Take *xy* as ground-line for elevation, and determine *p'*, and consider it as the centre of a sphere $\frac{3}{4}$ in. radius; then all planes tangential to this sphere must be $\frac{3}{4}$ in. away from the given point. Envelop this sphere with a cone whose base is on the vertical trace *vt* of the given plane, and whose base angle is 70°; also envelop it with a cone with its base on the horizontal plane, and base angle 65°. The circle of contact of each cone with sphere is shown by 1 2 and 3 4, their common point being *b'*; then *b'* is a point in the plane required. Join *v'b'*, *vb*, and produce to *c'c*, its trace; then a plane containing this line *v'b'vb*, and tangential to the base of the cone *fg*, vertex *v''*, is the required plane, marked by its scale of slope B. Or determine the elliptic base of cone vertex *v'*. The plane tangential to the bases of both cones will give the required plane. Or, again, the plane containing the line joining the vertices *v'v''*, and tangential to the base of the cone *fg*, vertex *v''*, will satisfy the given conditions.


v'
v''
2
3
4
b'
p'
1
vt
70°
65°
A
y
G'
-10
n
10
20
10
B
0
v
f
p
19
b
g
c

No. 44, 1887. **The traces of two planes are given, and *ab*, the plan of a line lying in one of them, MON; AB is one side of an equilateral triangle, of which the vertex C is in the other given plane. Complete the plan, and draw the elevation of the triangle.**

The traces MON, FGH, and the plan of the line *ab* are given. It is required to construct an equilateral triangle on *ab*, with its vertex C in the given plane FGH. We will commence by finding $a'b'$, and on its true length a_1b_1 construct the equilateral $\triangle$ $a_1b_1c_1$. Draw the $\perp$ c_1o_1; then c_1o_1 is the true length of the altitude. Now determine a plane *rst* bisecting *ab*, $a'b'$ and perpendicular to it, and find its intersection *dg* with the plane FGH; the vertex *c* must be in this line. Rabat this plane, carrying the point *o* and the intersection *dg* to the H.P., as shown at O and *dg*. Now with O as centre and radius c_1o_1, (the altitude of the triangle), cut off a length OC, OC equal to it, above and below the level of O, which are the heights of the vertices of two triangles fulfilling the given condition. The points CC rabatted back to *c* and c_1 on the intersection will be the required vertices. Join *cab*, and the upper triangle is complete in plan; its elevation is easily obtained, for vertex c' is on the elevation of the intersection $d'g'$. The other triangle fulfilling the given condition is shown at $a'b'c_1''$, *abc*, *i.e.* below the level of *ab*.


E
M
r
δ'
b
c₁
c'
X₁
True length
o₁
b'
G''
O''
a'
a₁
a'
c''
X
G
O
δ
d'
S
Y
c
a
o
b
H
N
c₁
d
c
o
G
G
Y₁
t

No. 44, 1888. **The three given planes, $vvhh$, $v_1o_1h_1$, $v_2o_2h_2$, together with the planes of projection, enclose a solid. Draw the plan of this solid when the face contained in the plane $v_1o_1h_1$ is horizontal.**

The sketch Fig. 1, is given here to half the scale of the drawing. Commence by determining the intersection of the planes, AC, $A'C'$ is the intersection of plane $V_1O_1H_1$ and plane VV, HH_1 and BD, $B'D'$ is the intersection of plane $V_2O_2H_2$ and plane VV, HH_1. The solid in question is bounded by the planes V_2H_2, V_1H_1, the plane VVHH between the intersections, and the co-ordinate planes. Make an elevation of the solid on X_2Y_2 perpendicular to the trace O_1H_1 of the face which has to be horizontal, as at $a'o'_1b'o_2c'd'$. Now, the face Ao_1c is in the vertical trace of this plane in this new elevation, as at $a'o_1'c'$, and if we take a ground-line parallel to this face, as X_1Y_1, we may say that that face is horizontal. Determine its plan in that position, as shown at $abcdo_1o_2$; thus corner a will be as far in front of X_1Y_1 as it was in front of X_2Y_2, and so on for every corner. If these points are joined the plan is completed. The plan might also be obtained by rabatting the solid about the trace O_1H_1 till the face AO_1C is in the horizontal plane; the former method is, however, more convenient and much simpler.

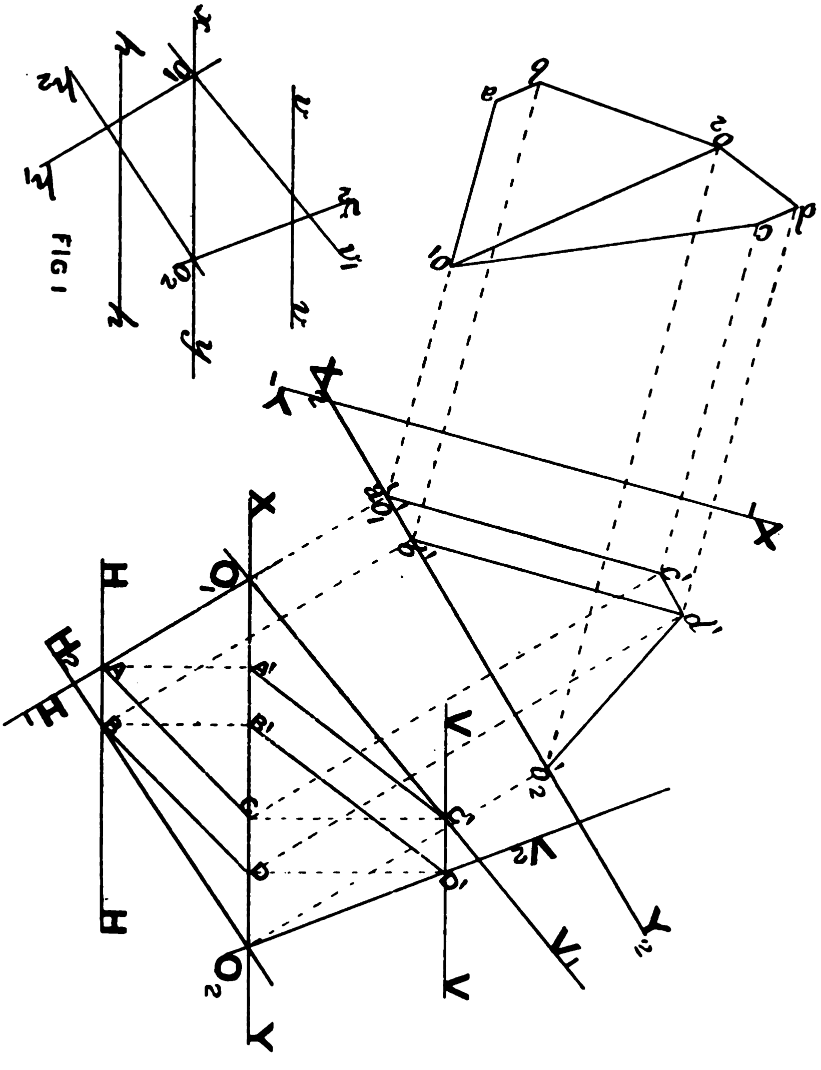
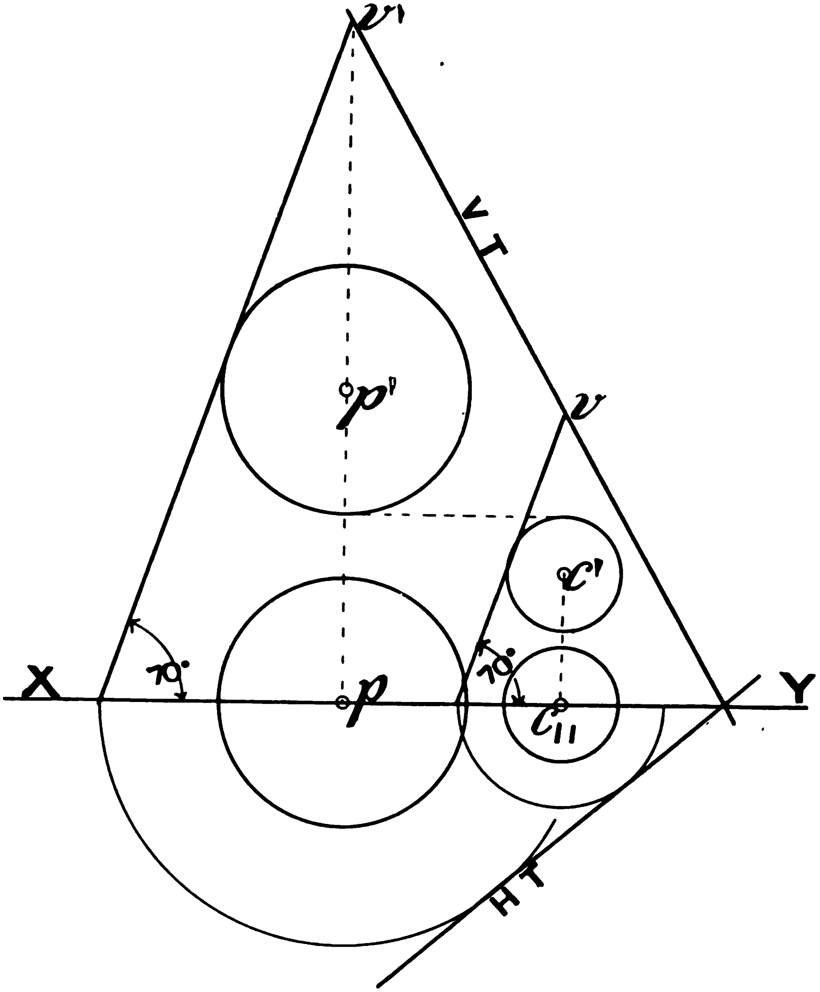


FIG 1

No. 58, 1891. **Two spheres are given, the index of the centre of the smaller being 11, and the lowest point of the larger being level with the highest point of the smaller. Draw a plane inclined at 70° to the horizontal plane, and touching both the spheres (unit = 0·1 in.).**

The spheres centres p and $c11$ are given. We may commence by determining the elevation of the spheres, as at $c'p'$; for convenience, take the ground-line XY through centre of spheres in plan.

Next determine a cone enveloping each sphere base angle of 70°, having their vertices at v and v'; the horizontal trace of the required plane will be tangential to the circular bases of these cones, as shown by HT. The vertical trace will evidently pass through vv', as shown by VT, for any plane containing the line joining the vertices of any two cones, and having its horizontal trace tangential to their bases, must be a plane tangential to both cones, consequently tangential to the given spheres.


v'
V T
p'
v
c'
70°
X
p
70°
c''
Y
H T

No. 55, 1894. **Through the given point pp' draw a line inclined at 20° and making 65° with the given plane.**

The plane is shown by its traces *mon*, and the point pp' is given.

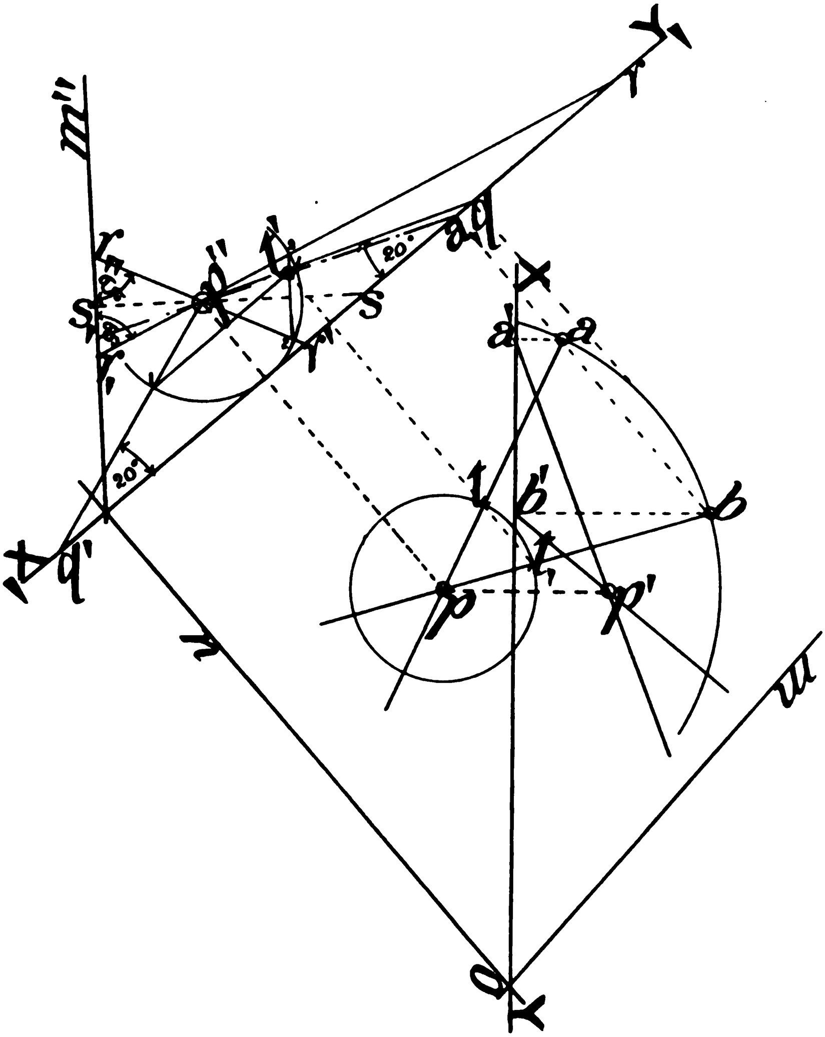
The required line will be the generator of intersection of two cones, one of which has its base on the horizontal plane, base angle of 20°, and its vertex at P; the other cone having its base on the given plane *mon*, base angle of 65°, and vertex at P.

The working of problems of this description is greatly facilitated by making a new elevation on a ground-line X_1Y_1 taken perpendicular to the given horizontal trace as shown, and sufficiently far enough away to be clear of the drawing.

Determine the vertical trace of the plane as shown at m'', and the elevation p'' of p. With p'' as vertex, draw the elevation $qp''q'$ of a cone with base on the horizontal plane, and base angle of 20°.

Also the elevation of the cone $p''r_1r_{11}$, with its base on the given plane and axis $p''s_1$ perpendicular to it, and base angle of 65°. Produce this cone to X_1Y_1 on the other side of the vertex, cutting X_1Y_1 in $r'r$.

The intersection of these two cones will be the required line, which may be found by drawing any sphere centre p'' cutting the generators of the cone $p''q'q$ in points 1, 1, and the generators of cone $p''r'r$, in points 2, 2. Join 1, 1 and 2, 2; their point of intersection t is a point on the generator of intersection. Join $p''t'$ and produce to cut X_1Y_1 in point a_1. Draw the projector $a_1\ ab$, cutting the circular base of cone $p''q'q$ in a and b. Then pa and pb are two lines fulfilling the given conditions. Their elevations are shown at $p'b$, $p'a'$.


m''
r
q''
t'
s
20°
a
b
b'
d
t
t'
p
p'
q'
x
y
n
m
o

No. 57, 1894. **Two lines, *ab*, *cd*, are given by their figured plans. Determine a line which meets *ab* and *cd*, and is parallel to and 2·5 in. from the plane whose scale of slope is given. (Unit = 0·1 in.)**

Take a ground-line XY parallel to scale of slope, and get the end view EF of the plane. Also get the elevations of AB and CD on the same ground-line. Now cut these lines with a plane parallel to the given plane and 2·5 in. from it. The points of intersection of lines and plane give the extremities of the required line. In the figure the parallel plane is shown in end view at *m'n'*. The plan of the line MN is at *mn*. The line MN must be parallel to and $2\frac{1}{2}$ in. from the given plane, because it is contained in a plane which satisfies that condition.

No. 57, 1890. **Three lines meet in an inaccessible point. The plans of two lines, *ab*, *cd*, are given, and also the plan *p* of a point on the third line. Describe a sphere in the pyramid of which the lines are edges. (Unit = 0·1 in.)**

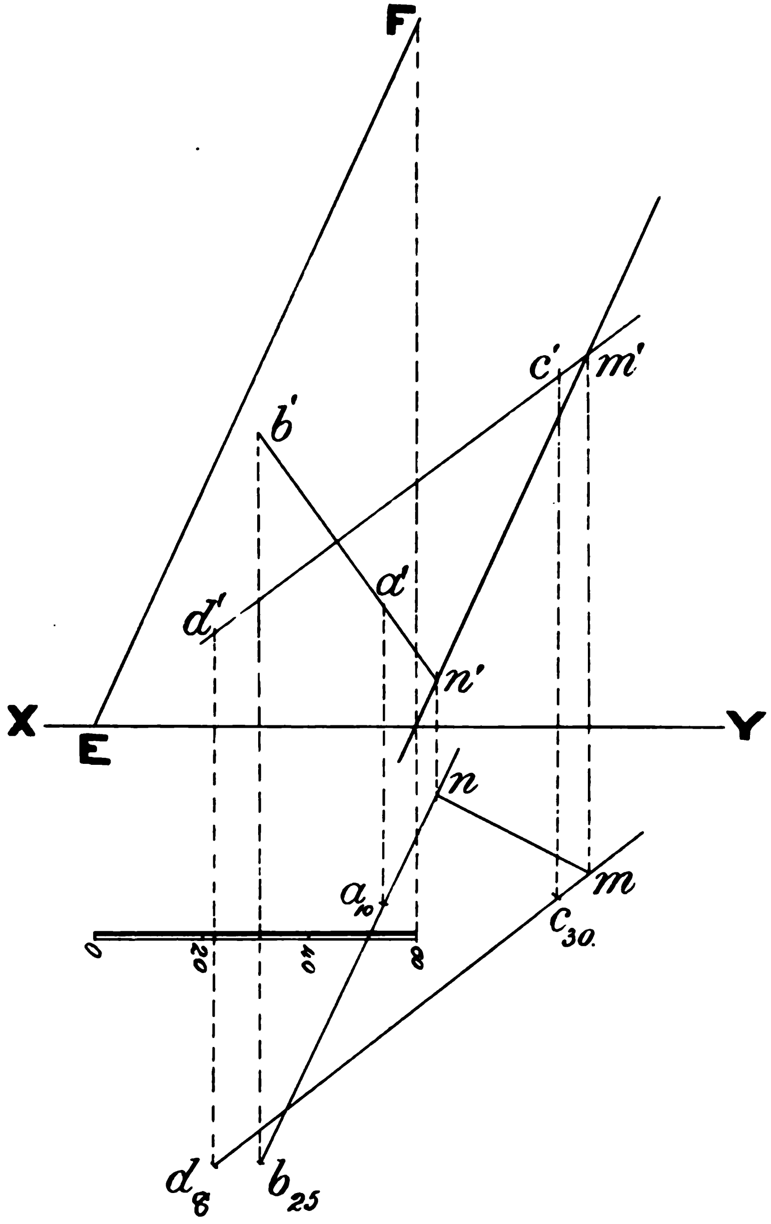
The given lines and point are indexed *a*5, *b*17, *c* not given, *d*9, *p*8.

The direction of the third line of which *p* is a point on it, will be readily determined by means of similar triangles.

Commence by joining the three points *bdp*, forming a triangle, and through *a* draw *av* parallel to *bp*, and *ac* parallel to *bd*, meeting *cd* in *c*; then through *c* draw *cv* parallel to *dp*, meeting *av* in *v*. The △ *avc* is similar to △ *bdp*, and *vp* is the third line required.

Their elevation may be determined by taking the ground-line in any convenient position, as at XY, and projecting each of the points whose heights are known, such as *b'd'p'* and *a*.

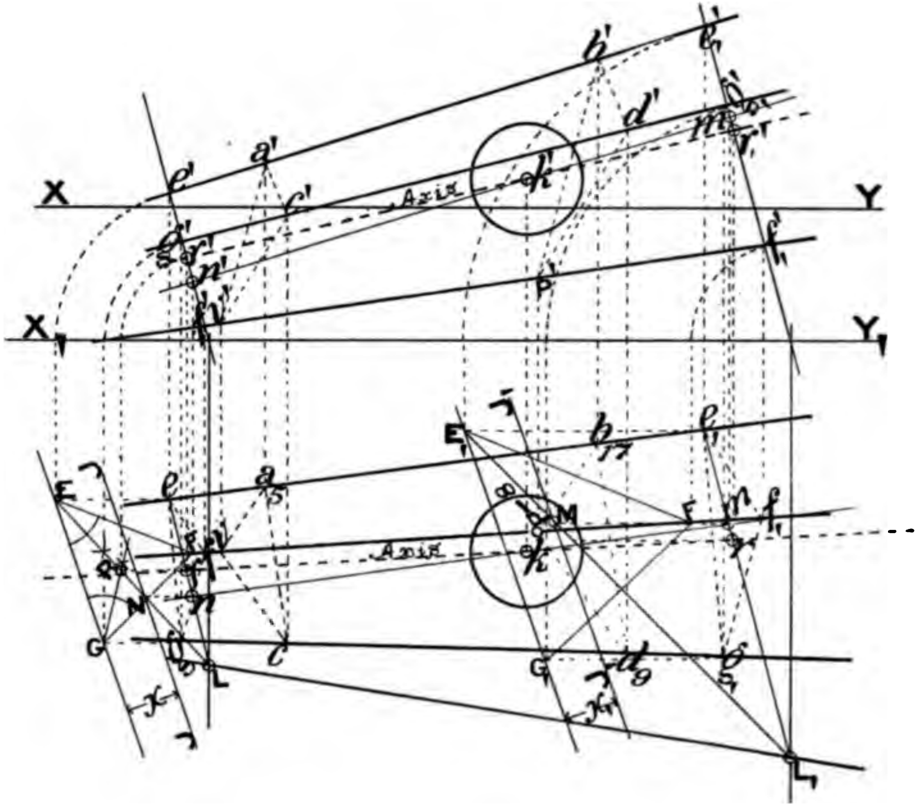
Now the heights of *c* and *v* have to be determined; these may be found in the same way as *v* was found in plan. Draw the triangle *b'd'p'*, and through *a'* draw *a'v'* parallel to *b'p'*, cutting the projector through *v* in *v'*. And through *a'* draw *a'c'* parallel to *b'd'*, cutting the projector


F
b'
c'
m'
a'
d'
n'
X
E
Y
n
a_{10}
m
c_{30}
0
20
40
60
d_8
b_{25}

through *c* in *c'*. Then *v'* and *c'* are points on the required lines. Join *c'v'*; $\triangle\, a'c'v'$ is similar to $\triangle\, b'd'p'$. Join these points respectively, and the elevation of the solid is complete.

Next we may proceed to determine the axis of the solid. To do this draw X_1Y_1 a little below XY, so as to be clear of the elevation. Make a section of the pyramid by a plane perpendicular to X_1Y_1, and (for convenience) as nearly perpendicular to its axis as possible, as shown by its traces *e'f'*L, cutting the edges of the solid in points *e'f'g'*, *efg*. Rotate this section about the H.T. till it meets the ground at EFG. Bisect any two of its angles, when the bisectors will meet in a point R; rotate the whole back to its original position, carrying point R with it to *r'r*. Then *rr'* is a point on the axis of the solid. Similarly, the point r_1r_1' at the other end of the solid is found. Join rr_1, $r'r_1'$; then rr_1, $r'r_1'$ is the axis of the solid.

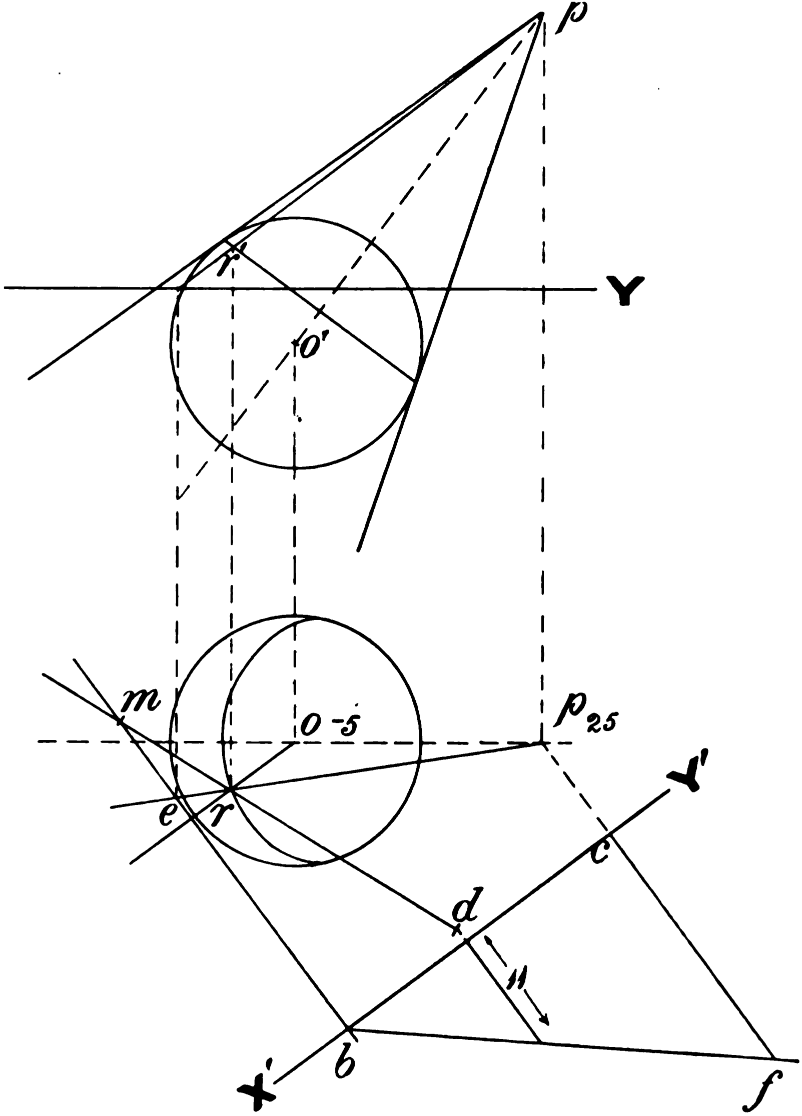
Now, to determine the radius of a sphere at any given point on the axis and inscribed in the solid, we must take a section of the solid through that point and perpendicular to one face, when the shortest distance from the point to that face will be the radius of the sphere at that point. However, another method is here adopted, it being much simpler in this case. We take a plane, JJ, J_1J_1, parallel to any face of the solid, as GE, at any distance x from that face (shown on the rabatted sections). This will intersect the plane that bisects the dihedral angle between the two contiguous faces *ge* and *fe* of the solid whose horizontal trace is LL_1, in a line *mnm'n'* which is parallel to the edge *aba'b'*. Now, this line will intersect the axis of the solid in a point, because it is contained by a plane which bisects the dihedral angle between two contiguous faces, and will also intersect the axis at a point distance x from the faces of the solid, because it is contained by a plane distance x from the faces. Consequently *kk'*, the point of intersection of *mnm'n'* with the axis, will be the centre of a sphere radius x, and inscribed in the solid.


X
Y
X₁
Y₁
e'
a'
c'
b'
d'
k'
Axis
m'
r'
e₁'
f₁'
p'
n'
E
e
a₅
F
Q
N
G
L
x
c
E₁
h₇
e₁
M
F
n
f₁
k
Axis
G
d₉
x₁
b₅
L₁

No. 58, 1890. **The given line *md* is the plan of a line which touches the given sphere, and lies in a plane also touching the sphere and containing the given point *p*. Obtain the indices of the points *m* and *d*. (Unit = 0·1 in.)**

Take a ground-line XY parallel to *op*, and draw the elevation of the sphere and the point *p*. Put a cone enveloping the sphere with its vertex at *p*′, and obtain the circle of contact of sphere and cone. Only half is found in plan, as more is not required.

The intersection of circle of contact and the plan of the given line will be the point where the line (and consequently the plane) touches the sphere. Join this point of contact *r* to the vertex *p*, and find the horizontal trace *e* of this line. This is one point in the horizontal trace of the touching plane. The radius of the sphere *or* at the point of contact must be normal to the plane, for a line perpendicular to a plane must have its projections perpendicular to the traces of the plane. Hence the horizontal trace of the touching plane must be perpendicular to *or*. Draw this trace *mb* and take an end view *bf* of the plane on the ground-line X′Y′. The point *m* in the given line happens to come in the horizontal trace, therefore its index is zero. The point *d* has an index of 11. There is a second possible position of the line, which would then touch the sphere at a point nearer the level of its centre.


p'
Y
r'
o'
m
o ·5
p25
Y'
e
r
c
d
11
b
f
X'

No. 54, 1894. **Determine a vertical plane cutting the given parallel planes in lines $1\frac{1}{4}$ in. apart.**

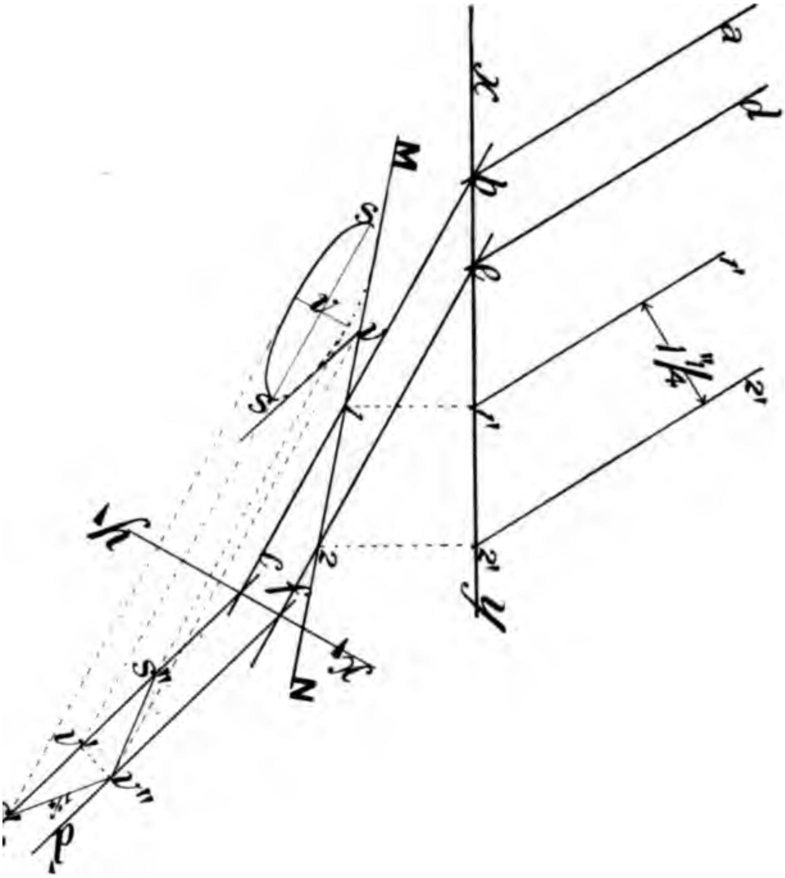
The planes *abc* and *def* are given. The required plane will be tangential to a cone, whose generator is $1\frac{1}{4}$ in. long, and having its vertex in one of the given planes, its base in the other, and its axis perpendicular to both.

Commence by making a new elevation, a' and d', of the planes on x_1y_1 perpendicular to their horizontal traces.

Draw the axis $v'v''$ in any convenient position, and complete the elevation of the cone, making the generator $v''s''$, $v''s'$, $1\frac{1}{4}$ in. long.

Determine its plan as shown at *vss*. Then the line MN, passing through the plan of the vertex and tangential to the base of cone will be the horizontal trace of the vertical plane required, cutting the traces of the given planes in points 1 and 2. Their elevations are at 1′ and 2′; and as the plane MN is vertical, its intersection with the given planes will be lines parallel to the vertical traces *a* and *d* respectively. Draw, therefore, the elevation 1′1′, 2′2′ of these intersections parallel to these traces, when their distance apart will be found to be $1\frac{1}{4}$ in.

Because the plane MN is tangential to a cone fulfilling the above conditions, the lines must therefore be $1\frac{1}{4}$ in. apart. It will be seen that another plane satisfying the required conditions may be found on the other side of the cone to MN.


x
y
a
d
b
e
M
N
s
v
1
2
1'
2'
1¼"
h
k
c
f
p

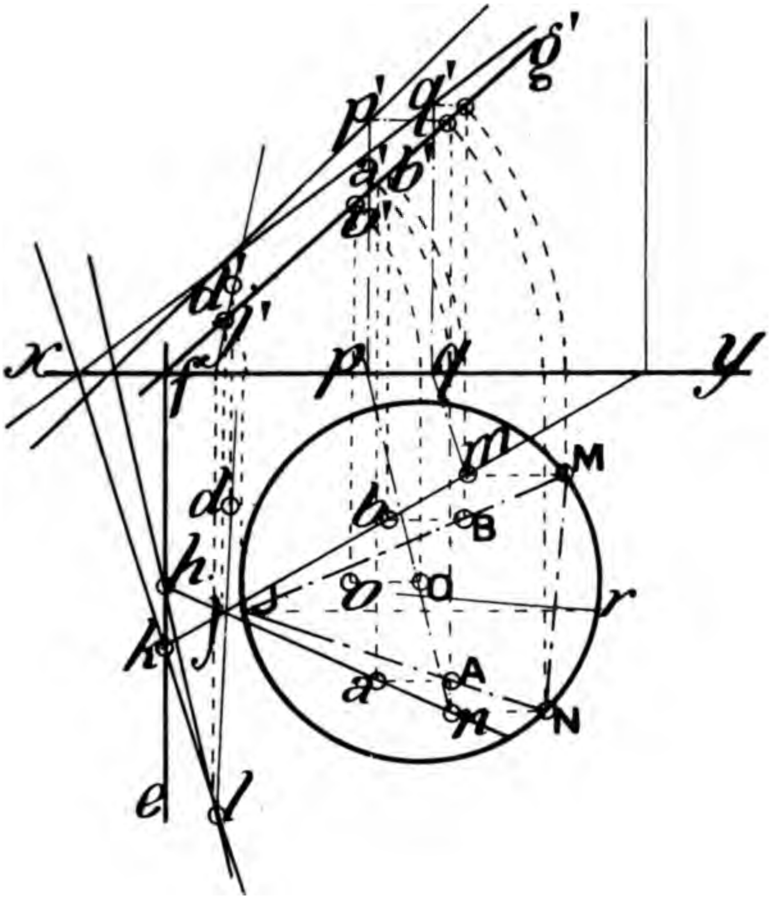
No. 53, 1896. **O, O′ are the projections of the centre of a circle of 1$\frac{3}{8}$ in radius lying in the plane *gfe*; *a* and *b* are the plans of two points A and B in the circle; *d* is the plan, and *d′* is the elevation, of a given point D. Draw the traces of two planes containing, respectively, D and B_1, and D and A; the chords of intersection of the planes with the circle are to meet at a point on the circumference, and the chord of the circle joining their other extremities to be parallel to AB.**

If we rabat the given plane *gfe* to the H.P., carrying the given points *a, o, b* with it to A, O, B, and then describe the circle centre O, radius 1$\frac{3}{8}$ in., as shown; then in the circle inscribe a triangle MNJ, having its base MN parallel to AB, and its other sides passing through A and B, and cutting the circle in the common point J: then the planes containing the given point D and each of the lines JM and JN respectively will be the planes required.

Rotate the plane back to its original position, carrying the point J with it to *j* and *j′*.

Now find the horizontal traces *l, h,* and *k* of the lines *dj, nj, mj,* respectively. It will be seen that *dj* is common to the two planes. Join *lk* and *lh*; then *lk, lh* are the horizontal traces of the required planes. Now their vertical traces are easily determined, thus: Draw *np* parallel to *lh*, and *mq* parallel to *lk*, meeting the vertical plane in *p* and *q* respectively*: then *np, mq* are levels on the planes. Erect *pp′, qq′* perpendicular to *xy*, equal in height to *n* and *m* respectively: then *p′* and *q′* are points in the vertical traces of the planes. Hence *lhp′* and *lkq′* are the traces of the planes satisfying the given conditions.

* Points *n* and *m* lie between *b* and *g* on line *fg′*.


g'
p'
q'
t'
a'
b'
e'
d'
f'
x
f
p
q
y
m
M
d
b
B
h
o
O
r
J
k
j
a
A
n
N
e
l

No. 54, 1895. **Three lines are given: AA′, AB, CC′, and a point *aa′* on AA′. Determine a point *bb′* on B such that the line *ab*, *a′b′* shall be 1½ in. in length, and sloping downwards from *aa′*.**

Find on line CC′ the projections of a point *cc′* such that the sum of the real distances of *cc′* from the points *aa′* and *bb′* shall be 3½ in.

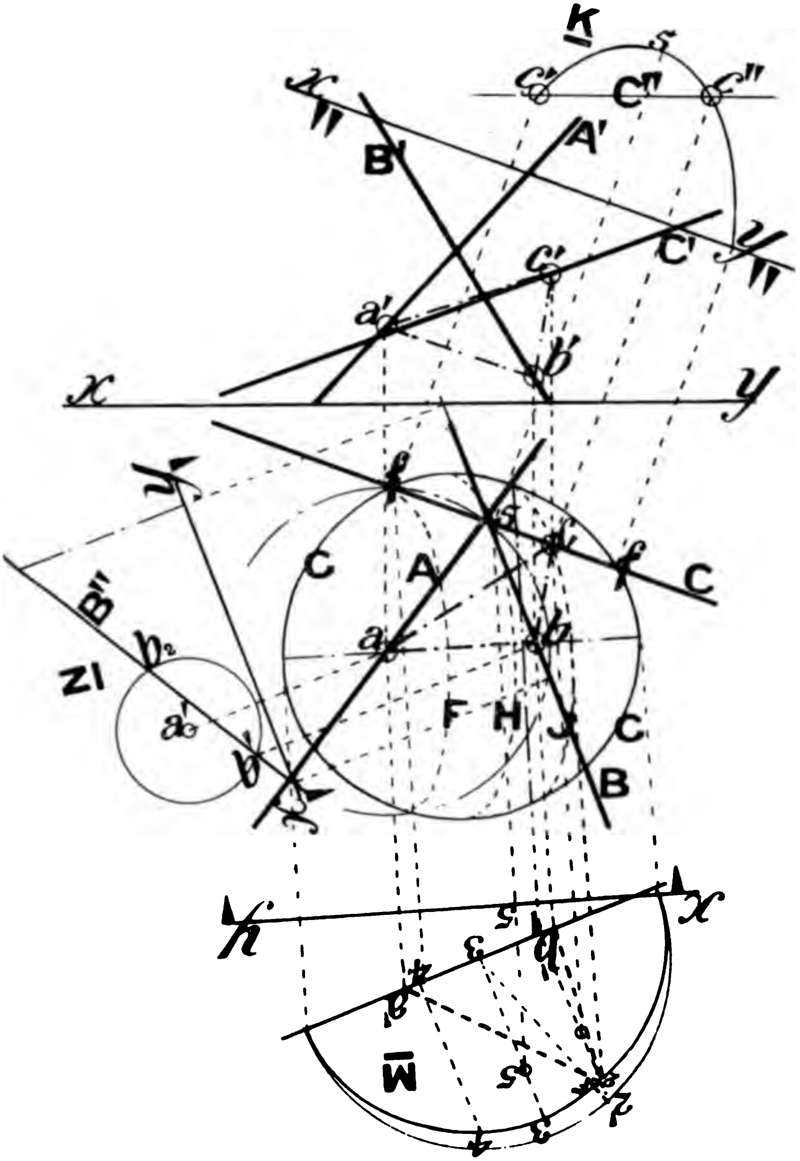
The point *bb′* on BB′ will readily be found, if we consider *aa′* as the centre of a sphere 1½ in. radius. The lower point of intersection of line B and sphere will be the required point *b* 1½ in. from *aa′*. Draw a ground-line x_1y_1 (Fig. $\overline{\text{N}}$) parallel to line B, and make an elevation of line B at B″. Now consider the line B as the horizontal trace of a vertical plane which would cut the above sphere in a small circle, shown in elevation centre *a′*. This circle and the line B will lie in the same plane, their intersecting points *b′*,*b*2 are the points where the line interpenetrates the sphere, both of which are 1½ in. from *aa′*. It will be seen that *b*2 is higher than *a′*; we must, therefore, select *b′* because it is lower than *a′*, as, according to the question, the line *ab* must slope downwards from *aa′*. Now project *b′* back to line B in plan at *b*, and to the elevation on the given *xy* at *b′*.

Then *a′b′ab* will be the required line 1½ in. long and sloping downwards from *a*.

Next make an elevation of *ab* on a ground-line x_1y_1 (fig. $\overline{\text{M}}$) parallel to *ab*. And on it construct any triangle, as *ab*2; the sum of its two sides $a_1 2$ and $b_1 2$ shall be 3½ in.

The locus of its vertex 2 will be an ellipse, a_1 and b_1 being the foci. (The sum of focal distances equals the length of the major axis.) Describe a portion of the ellipse, given the foci a_1 and b_1, and the sum of the focal distances equal 3½ in.

Now imagine this ellipse to rotate about its major axis, when it will generate a surface of revolution called a spheroid, its sections parallel to its major axis being ellipses, and parallel to its minor axis being circles.


K
5
C''
c''
x''
y''
B'
A'
a'
c'
C'
b'
x
y
f
s
f
C
A
C
a
b
B''
b
Z1
a
b
F
H
J
C
B
h
x
5
3
d
4
M
5
3
4
2

Every point on its surface will evidently have the sum of its distances from a_1 and b_1 $3\frac{1}{2}$ in. It is therefore the surface locus of a point, moving in such a way that the sum of its distance from a_1 and b_1 is constant and equal to $3\frac{1}{2}$ in.

The plan of the solid is shown at GG. On x_{11}, y_{11} (fig. K), parallel to line C, make a new elevation of line C at C″, and also the true shape of section made by the imaginary vertical plane containing the line C. This section is obtained by first determining the elliptic plan of the circular sections of the spheroid at 44, 33, 12 (fig. M), shown in plan at FHJ.

Then the heights above xy of every point in the section may readily be found in fig. M, such as 55, and these heights transferred to fig. K as shown, the curve drawn through these points will be the true shape of section * of the solid made by the vertical plane containing the line C.

The points c' c'' of interpenetration of the line C with this solid, will be points satisfying the conditions given; because the line C and true shape of section are in one and the same plane, the latter being, as already explained, on the surface locus of a point moving such that the sum of its distance from a_1 and $b_1 = 3\frac{1}{2}$ in.

The plan of c'' is at c; then the triangle abc, $a'b'c'$, will have its angles on the three given lines, and also the side ab, $a'b' = 1\frac{1}{2}$ in. (and sloping downwards from aa'), also the sum of the sides ac, $a'c'$, bc, $b'c' = 3\frac{1}{2}$ in. as required.

It will be noticed that two solutions are possible, for C′ and C″ are two points satisfying the conditions given.

---

* Part only of the section is here drawn; the lower half is not required.

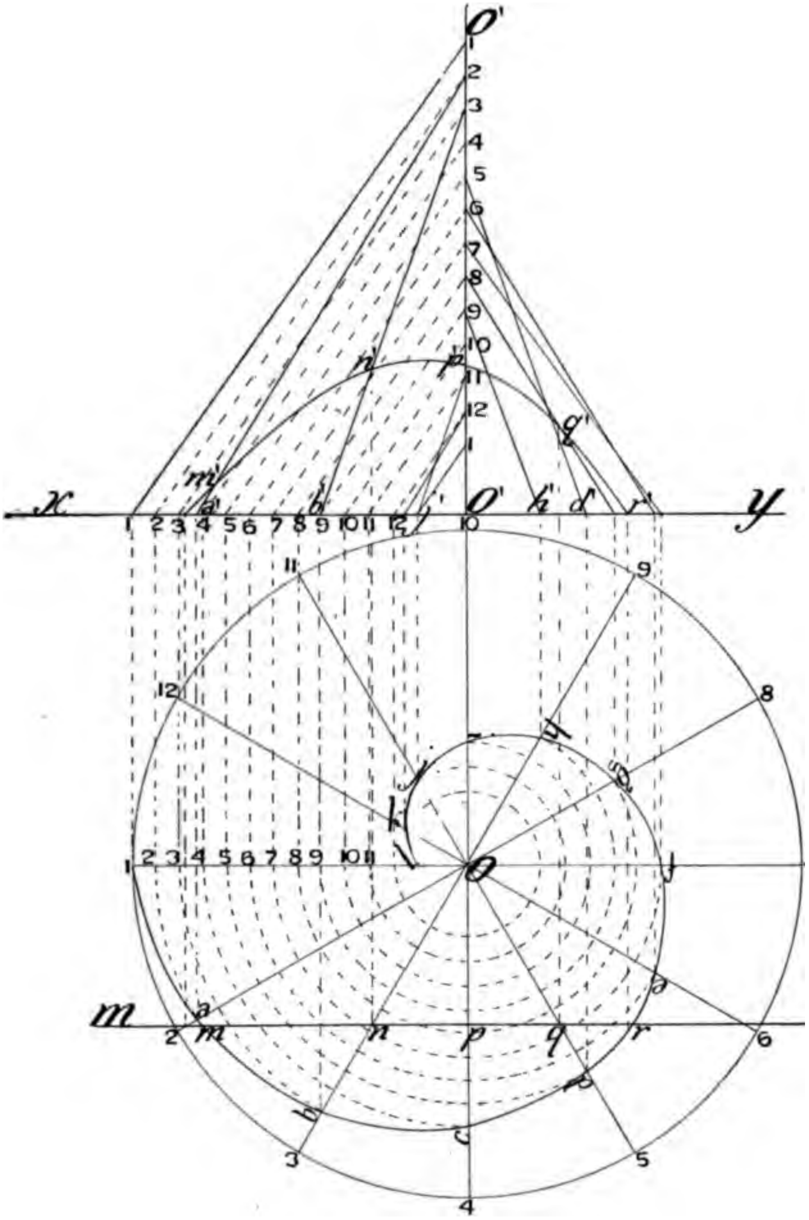
CHAPTER V.

SECTION OF SURFACES AND SOLIDS.

50, 87. **A surface is generated by a line passing through a fixed vertical axis and making a constant angle with it of 35° while revolving round it at a uniform speed. During a complete revolution, the point of intersection of the generating line and the axis moves $2\frac{1}{2}$ in. along the latter. Draw the section of the surface so generated by a plane parallel to and 1 in. from the axis.**

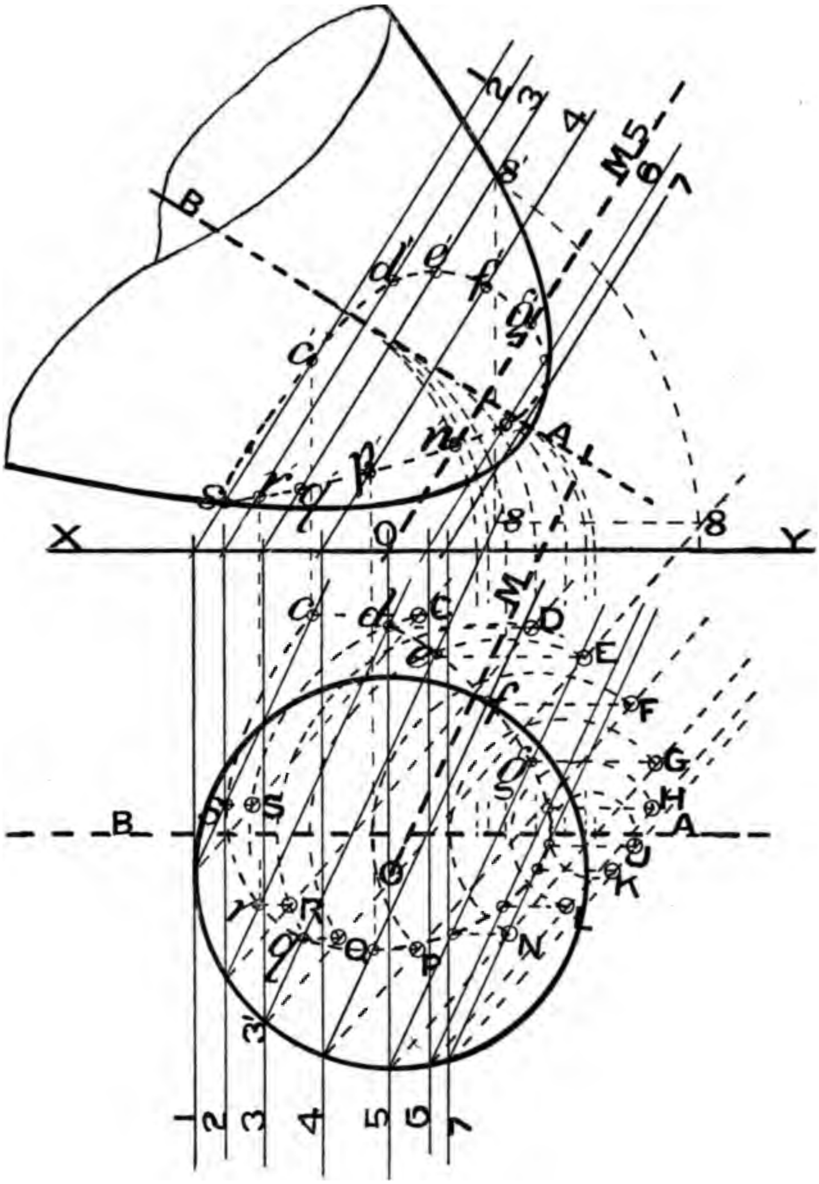
The line in question will diminish uniformly in length while revolving round the vertical axis, and the locus of its trace on the horizontal plane will be a spiral. Let *o o′o′* be the vertical axis. Draw *xy* perpendicular to it, and draw any line meeting it at an angle of 35° such as 1 1, forming the right-angled △ 1 1 0′. Next measure down a distance of 2½ in., through which the line travels on the axis, and divide it into any number of equal parts as shown, 1 to 12. Parallel to the hypothenuse 1 1, from each of the points of division as shown draw lines 2 2, 3 3 . . . 12, 12. Now these will represent the length of the line at the various stages of revolution. Then, with *o* as centre, radius 0 1, draw a circle and divide it into the same number of equal parts as the axis, 0′0′, such as 1, 2 . . . 12.

The plan of the lines will coincide with the radial lines 0 1, 0 2 . . . 0 12 at all points; their plan lengths are seen in the elevation on *xy* such as 0′1, 0′2, 0′3 . . . 0′12. Now project each of these points to the diameter 0 1; then, with 0 as centre and radius 0 2, draw the arc 2*a*, cutting the first radius in point *a*, and 0 3 cutting second radius in point *b*, 0 4 cutting third radius in point *c*, and so on for each till you arrive at the point *l*. Now this will be the plan of the surface generated. Its elevation may be readily drawn by projecting each point to *xy*, such as *a* to *a′*, *b* to *b′*, *c* to *c′*, *d* to *d′*, etc.; join 2 to *a′*, 3 to *b′*, 4 to *c′*, etc., for every generator which will complete the elevation of the twisted surface. The section plane drawn parallel to *xy* and 1 in. away from the axis *o*, cuts these lines which generate the surface at points *m*, *n*, *p*, *q*, *r*. The curve obtained by joining the elevation of these points, such as *m′*, *n′*, *p′*, *q′*, *r′*, will be the section of the surface made by the plane *mn*.


o'
1
2
3
4
5
6
7
8
9
10
11
12
1
n'
p'
q'
m'
x
a'
b'
j'
o'
k'
d'
r'
y
1 2 3 4 5 6 7 8 9 10 11 12
10
11
9
12
8
1 2 3 4 5 6 7 8 9 10 11
o
m
2
m
n
p
q
r
6
a
b
c
3
4
5

No. 51, 1889. **A paraboloid of revolution (given by its elevation and plan of its axis) intersects an oblique cylinder, the axis OM of which is given, as also the plan of its base. Determine the projection of its intersection. (It will be noticed that the axis of cylinder is parallel to planes of circular sections of the paraboloid.)**

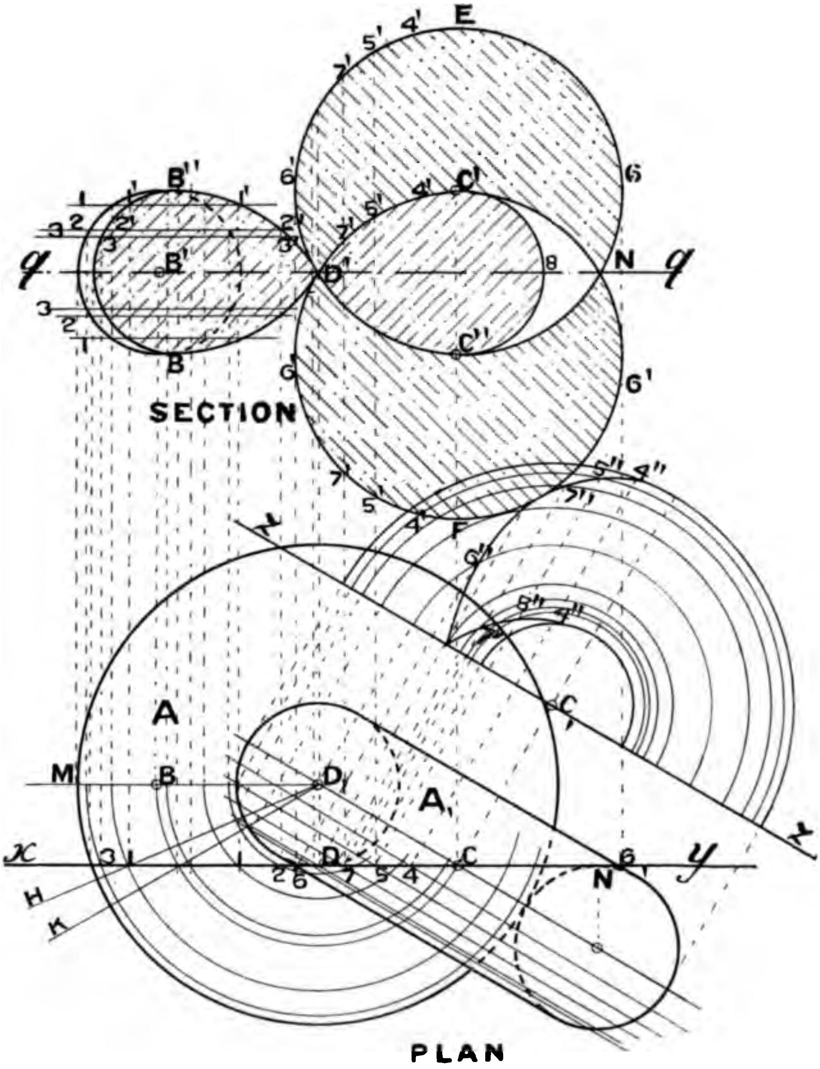
The paraboloid as given is represented by its axis AB, and a portion of its elevation shown by the curve, and by the plan of its axis AB; the axes O'M', OM and the circular base of the cylinder are also given. This question has a complicated appearance, but after all it is extremely simple. Remembering that all planes cutting the paraboloid perpendicular to the axis will cut it in circles, and the cylinder will be cut in generators parallel to its axis under the given conditions; take a series of planes to cut the solids as above described, such as 1 2 3 . . . 7; and rabat each circular section and each generator to the ground. The points in which the circles intersect the generators of the cylinder will be points on the curve, and these thrown back to their original position will give the projection of the intersection. Let us consider one of the cutting planes, such as 3; it cuts the circular base in point 3_1. The plan of this generator is $3_1 8$; rabat this to the ground at $3_1 8_1$, shown in dotted lines; rabat also the centre of circular section to the ground at 3″. With 3″ as centre and radius = 3″8′, describe a circle in plan, crossing the generator in points Q and E; rabat these points back to the plan of generator in *q* and *e*, and so on for any number of points, such as C, D . . . R, S to *c*, *d* . . . *r*, *s*, which are points in the curve of intersection of the two solids. It is only necessary to rabat one generator such as 3 8 to the H.P., for all other generators will rabat parallel to this, and all points will rabat back to plan of generator parallel to *xy*, so that it is unnecessary to throw the points up to their planes. The back generators are not cut by the intersections.


1
2
3
4
M5
6
7
8
B
d
e
f
g
c
s
A
n
p
q
r
s
X
O
8
8
Y
M
c
d
C
D
E
e
l
f
F
G
H
A
B
S
s
J
K
R
r
q
O
P
N
L
l
3
1
2
3
4
5
6
7

No. 49, 1887. **The given figure represents the plan of two equal anchor rings interlocking. Make a vertical section on the line *xy*, distinguishing the sectional parts of the two rings.**

The figures given are marked A, A_1 in plan, *xy* being the plan of section-plane. The ring A is horizontal, and the ring A_1 is vertical.

Let us commence with the vertical ring A_1. It will be seen that the plane *xy* cuts through the centre C of the ring, and leaves its surface at point 6 and 6_1. Draw a line *qq* in any convenient position on your paper and parallel to *xy*, sufficiently far away from the plan so that they may not fall one on the other, and let line *qq* be the centre line of the solids. As the section only is required, we may erect a perpendicular from C, as CF,C″C′ E. Make FE equal to three thicknesses of the rings which the plane cuts. C′C″ will be the contact points of the rings on the interior surface. Draw a vertical projector 6 6 to meet the horizontal projectors from C′ and C″. It will be seen that the distance C′6 equals the thickness of the rings; and that the circles struck with C′ as centre and radius equal to C′6 will be the true shape of section of the upper exterior half of ring A_1, combined with the lower interior of the bottom half. Similarly, with C″ as centre, and same radius, describe a circle cutting the first circle in points D′N. This will be the true shape of the lower exterior half of ring A_1 combined with upper interior of the top half. The section of the ring, therefore, is included between parts of these circles; thus the upper section is included between the arcs D′C′N, D′EN, and the lower section between the arcs D′C″N, D′FN, which are shown scored. The portion included between C′D′C″ is the section made through the horizontal ring A. Its corresponding part on the left of D′ may be readily obtained by taking BB″ as centres, as far from D′ to the left as C′C″ is to the right, and radius as before. Draw the arcs B″D′, BD′. To complete the section of the horizontal ring to the left of BB″, draw a circle struck with B′ as centre and radius equal to BM (in plan), and take a series of horizontal cutting planes, as 1 1, 2 2, etc., giving circles in plan, which


E
5'
4'
7'
6'
6
B''
C'
4'
5'
7'
1
1'
1'
3
2
2'
2'
3'
3'
q
B'
D''
8
N
q
3
2
1
B
C''
6'
6'
SECTION
7'
5'
4'
F
5''
4''
7''
6''
5''
4''
7''
x
C,
A
M
B
D,
A,
x
3
1
1
2
6
D
7
5
4
C
N
6
y
H
K
PLAN

circles cut the plane xy in points 1, 3, etc. These points are then projected back to their respective planes, thus point 1 to level 1_1 at 1′, 3 to 3′, etc., and similar points found below line qq; the curve B″ 1 2 3B will complete the section of the ring on the left half. The portion on the right of C′C″ will be found in the same way, by projecting horizontally across from each point to a distance on the right of line C′C″, as the points were on the left of line BB″. The scored portion included by the curve D′B″2′3BD′, D′5′4′C′SC″D′ will be the complete section of the horizontal ring A.

However, another method is shown which is applicable to all cases, whether the plane of section pass through the centre C or not. Thus on line zz make an elevation of ring A_1, and take a series of vertical sections through it, which cut it in circles shown on this elevation, their intersections with the plane in plan, such as 6 7 5 4_1 projected to their respective circles in the elevation, are 6″7″5″4″ on the upper side, and 7″5″4″, etc., on the lower side. These projected to the elevation of the section through xy to distances above and below qq, such as 6′7 4′5′, equal to their heights above zz. These points, carefully joined, will be a true circle as before, *i.e.* for the upper and lower half respectively.

Again, the section through the horizontal ring may be found by taking points in the horizontal sections where they cut the circles struck with centre B as before; thus, section 1 1 cuts the ring A in two circles in plan, intersecting the plane xy in points 1, 1. These points, projected to 1′, 1′, will be points on the curve. Similarly, the plane $2_1 2_1$ will have two circles in plan, cutting the plane xy in points 2, 2; also the plane 3 3 cutting plane xy in points 3, 3, and these projected to 2′2′, 3′3′ will be points on the curve. The joining of these points will form the curve of section, and will be found to agree with the section already obtained. The curve of section of the right side of the ring will be similar, and projected horizontally to the right of D′, a distance away equal to the distance they originally *were to* the left.

90, 1. **Two non-intersecting lines, AB and CD, are given. A surface is generated by a line 3 in. long, moving with its extremities on the given lines. Determine the section of the surface by a plane perpendicular to the shortest distance between the lines, and passing through its centre point (unit = 0·1 in.).**

The thick lines *a*7, *b*30, *c*10, *d*30, Fig. 1, are given.

The surface generated by the moving line, and also the section made through it, will be more readily obtained if we so arrange the lines that one of them be in a vertical position, and then determine their elevation on a plane parallel to the plan of the other.

We may commence by determining their elevations on XY parallel to the plan, of *cd*, Fig. 1.

Now take X_2Y_2, perpendicular to *c′d′*, and determine plans (Fig. 2) CD and AB. Note CD is vertical; its plan therefore is a point, and will be as much in front of X_2Y_2 as it was in front of the original XY. The same for A and B.

Next determine their elevations on X_1Y_1, parallel to the plan of AB. In this position we see both lines in their true length; and here also the surface is most readily drawn, it being contained between the vertical planes containing lines AB and CD, and the elevation of every line generating the surface from C′D′ to A′B′ will be of equal length, and will be equal to the radius of the circular section made by the vertical plane containing AB of a sphere 3 in. radius struck with centre on C′D′ at different heights. But we need not draw the sphere in all its positions, for we may draw its plan as shown, cutting the plane of AB in *p* and *e*. It will be seen that all spheres having their centres on CD will have the same plan, and every section of the spheres made by the plane of AB will be of the same radius, and in every position, of sphere the line AB will cut the section of sphere in a point 3 in. away from CD. We may then take with our compasses the length *mn* or *mp* (the radius of sections made by the plane of AB) as radius, and in all possible positions, with centre on CD, describe arcs cutting the line AB in points which, on being joined to the centre, represent a line 3 in. long, having its extremities on both the given lines. Thus, with 1 on CD as centre, cut off length 1′ 1″ on A′B′ to the right and left of C′D′, with 2 cut off 2′ 2″, with 3 cut off 3′ 3″, and so on for any number of points; these points, being joined to their proper centres respectively, will be the elevation of the surface so generated on the plane, of which X_1Y_1 is parallel to the plane of AB, C being in a vertical position.

Next determine the plan of the surface thus: All points on A′B′ will have their plans on AB, as *e*, *f*, *g*, 1, 3, 4, 5, etc. Join each point to the plan of the line CD, and this will complete the plan of the surface. It will, however, be noticed that two generators have the same plan in several cases, such as 2 2, 2*g*, 3 3, 3*f*, 5 5, 5 5, etc.

Now, since the line CD is vertical, the perpendicular distance between the two lines must be CD*m*, and the plane bisecting it must therefore be vertical, and also parallel to the plane of AB, shown by its horizontal trace WW, cutting through all the generators in plan as shown at *r*, *s*, *k*, etc. Now, *r* is the plan of two points of section of two generators, for 1CD is the plan of the two generators 1 1′, 1 1″. Project, therefore, from *r* to *r*′ and *r*″, cutting the generators 1 1, 1 1; *s* to *s*′ and *s*″, cutting generators 5 5′, 5 5″; *k* to *k*′ and *k*″, cutting the generators 7 7′, 7 7″, and so on for every point: these points, carefully joined by a curved line, will be a true ellipse, which is the section of the surface in this position. But we have to throw the section back to the "original" position of the given lines. To do this, determine the major and minor axis E″F″, G″H″ of the ellipse, their plans being on the plane of section at EFGH, and as it is the section only that is required, we need not trouble about any part of the surface generated. Then determine the elevation of the axis E′F′G′H′ on X_2Y_2, remembering that each axis is at the same level above the horizontal plane in each case; thus *xy*

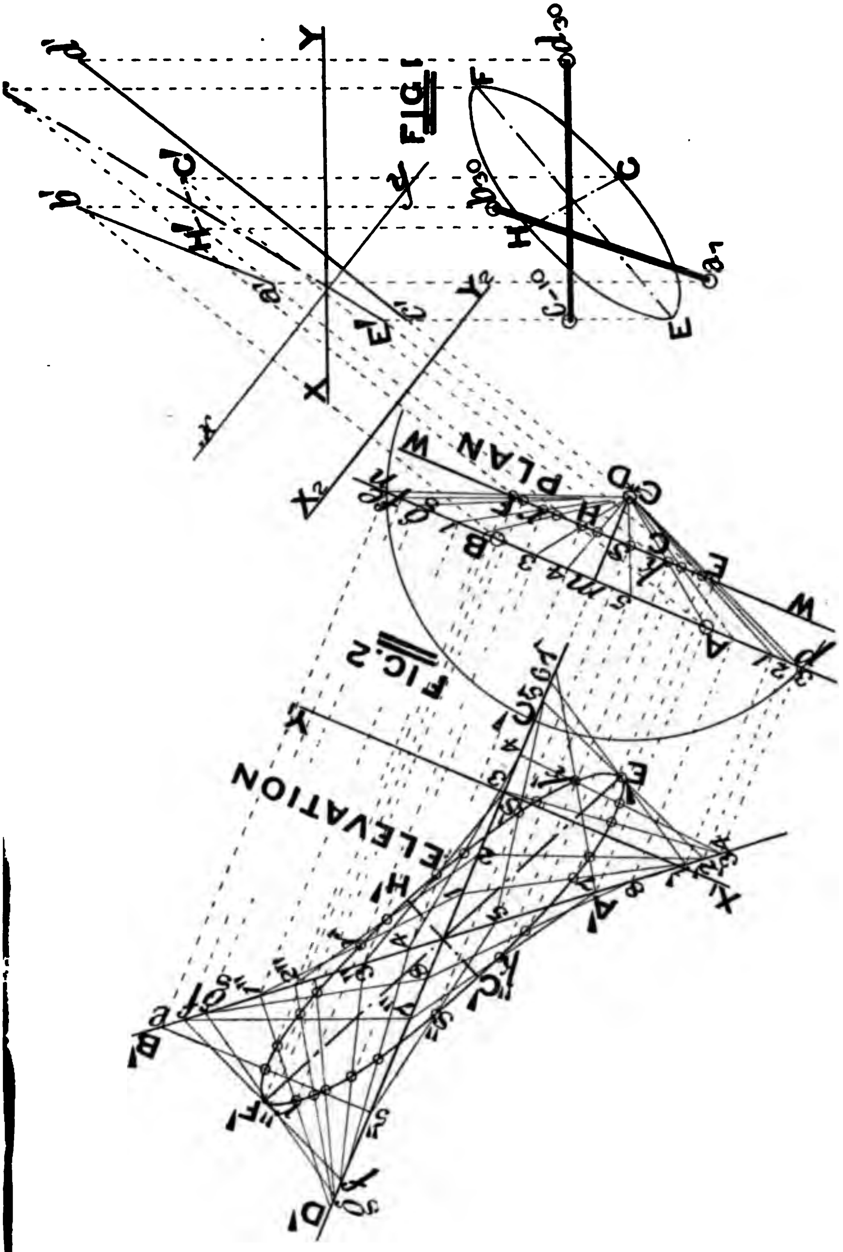
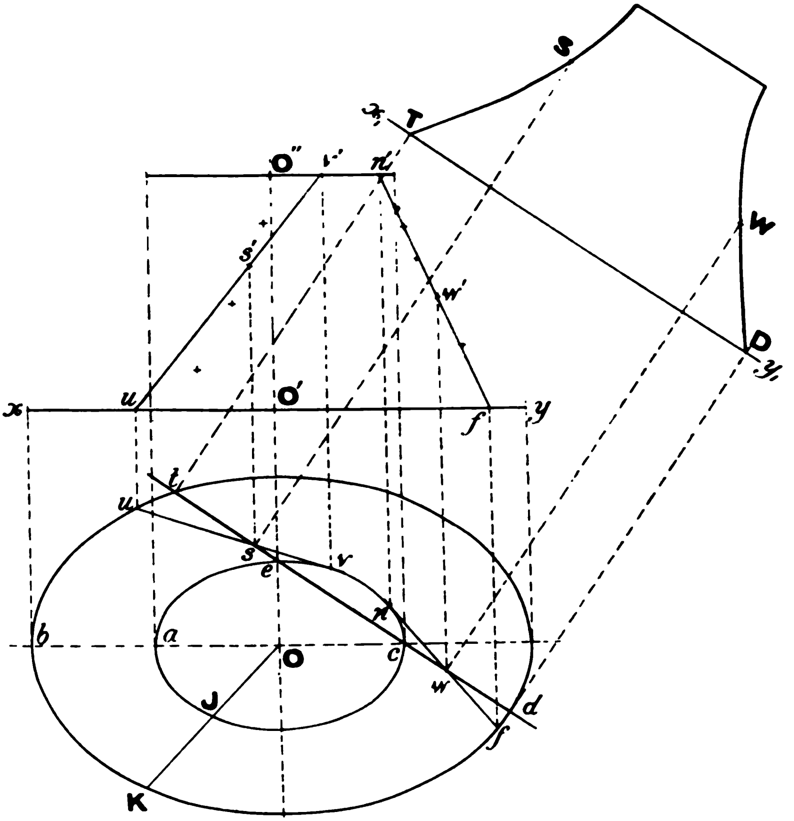


FIG. 1
FIG. 2
PLAN
ELEVATION
X
Y
W
W
F
E
C
H
G
D
B
A

is drawn parallel to X_2Y_2 at the same level as X_1Y_1, hence points F′, G′, H′ are as much above xy as they are above X′Y′ (Fig. 2), and E′ below xy as much as it is below X_1Y_1. Now determine the plan of the axes relatively to the first xy; they will be as much in front of it as they are in front of X_2Y_2, as at E, F, G, H. Draw the major and minor axes, and complete the ellipse as shown in plan, Fig. 1, which is the required section of the surface so generated by the line 3 in. long, with its extremities on the given lines.

No. 37, 1882. **Draw two concentric similarly situated ellipses, the axes of the larger ellipse being 3¾ in. and 2½ in., and the smaller respectively one-half of these dimensions. The larger ellipse is in the horizontal plane, and the smaller 1¾ in. above it. A surface is generated by a line which terminates on the two ellipses, and is always tangent to the smaller ellipse in plan. Determine the section of this surface by a vertical plane passing through one extremity of each axis of the smaller ellipse.**

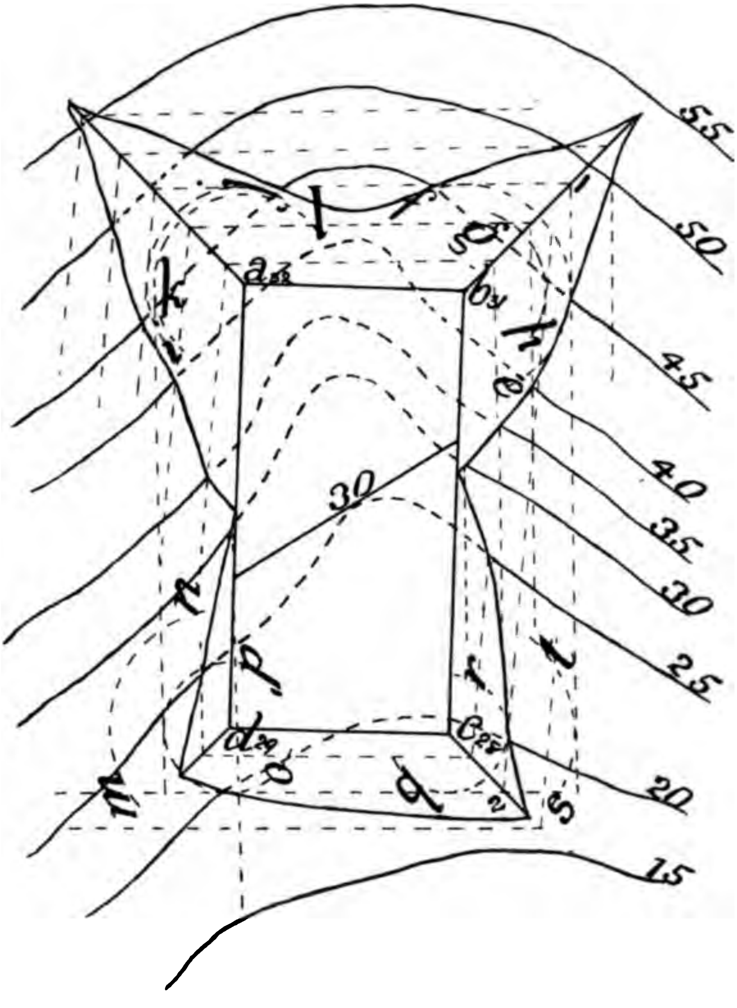
Construct the larger ellipse by any of the ordinary methods, and the smaller one may be obtained by drawing a series of radii vectores from the centre O and bisecting them. The points of bisection, such as J in OK, lie on the smaller ellipse. The elevations of the ellipses are xy and lz. Through e and c draw the horizontal trace $tecl$. Draw any line, such as nf, which will be a tangent to the smaller ellipse and terminate in the larger one, at the same time cutting the line tl in w. The elevation of nf is $n'f'$, and the point w' is obtained by projecting vertically from w. Take a new ground-line x_1y_1 parallel to tl, and project up the point w, and set up the height of W above x_1y_1 equal to that of w' above xy. W is a point on the section of the surface; in the same way S is a point on the section of the surface. About [illegible]ints on each curve will be sufficient. They are sh[illegible]ion.


S
x_1
T
W
D
y_1
O″
v′
n
s′
w′
u
O′
x
y
f
t
u
s
v
e
n
b
a
c
O
J
w
d
f
K

No. 52, 1888. **A portion of ground contoured at 5-ft. levels is given. A plane surface *abcd* has to be formed partly by cutting and partly by embankment. The side slopes are 45° in cutting and embankment. Draw the contours of the completed earthwork (scale 1 in. = 20 ft.).**

The contour-lines given are marked 15 . . . 55, and the plane surfaces *a*32, *b*31, *c*28, *d*29 also are given; these contour-lines are plans of sections cut by horizontal planes through a hilly surface at every 5 ft. in height, and by means of these contours the inequalities of the surface are correctly represented.

The slopes, of course, are plane surfaces, and all levels on them are straight lines, and parallel to each other at distances of 5 ft. apart in plan, since the dihedral angle is 45°, *i.e.* 1 vertical and 1 horizontal. We may begin by taking any contour-line to start with as 45 ft. Take *b* as centre whose level is 31 ft., that is, 14 ft. lower than the 45-ft. contour, and radius 14 ft., describe the circle *ef*. Also *a*32 is 13 ft. lower than the 45-ft. contour; therefore with *a* as centre, radius 13 ft., describe the circle *ij*: the plan of contour 45 on the plane slopes will be a tangent to these circles. A series of contour-lines drawn parallel to this, and 5 ft. apart in plan, cutting the original contours in points to the right and left, will, on being joined, give the contour of the cutting with the original surface. Now take the right-hand side. The level 45 ft. must be a tangent to the circles struck with *b* and *c* as centres and radius equal to their difference of level with the 45-ft. contour—the circle *fe*, radius 14 ft., and circle *st*, radius of $45 - 28 = 17$ ft. Their tangent meets the same level in back slope in point 1; then *b*1 is the intersection of the back slope and the right-hand slope. A series of levels, at 5 ft. apart will cut the originals as before in points where the cutting reaches the surface. The left-hand side is obtained in an exactly similar way. In the embankment on the right-hand side,


55
50
45
40
35
30
25
20
15
l
f
g
k
a53
b
h
i
e
30
n
p
r
t
d29
c25
o
q
2
s
u

the 20-ft. contour will be a tangent to circles struck with *b* and *c* as centres, with radii 11 ft. and 8 ft. respectively; and on the near side the 20-ft. contour will be a tangent to circles struck with *c* and *d* as centres, and radii 8 ft. and 9 ft. respectively, meeting the other level in point 2: then *c*2 is the intersection of the embankment surfaces. The left-hand corner is found in exactly the same way. Levels parallel to these and 5 ft. apart will cut the original levels in points which, on being joined, give the contour surface where the embankment meets the original surface.*

Another method may be adopted by taking a ground-line through *b* and perpendicular to *cb*, and drawing the vertical trace of a plane at an angle of 45°. Similarly, with a ground-line through *c* and perpendicular to *bc*, draw the vertical trace at an angle of 45°. On these determine levels, such as 45 ft. – 31 ft. and 45 ft. – 28 ft. respectively; their plans, being joined, will give the plan of the level, and so on. A series of levels 5 ft. apart, as before, will determine the crossing-points with the originals.

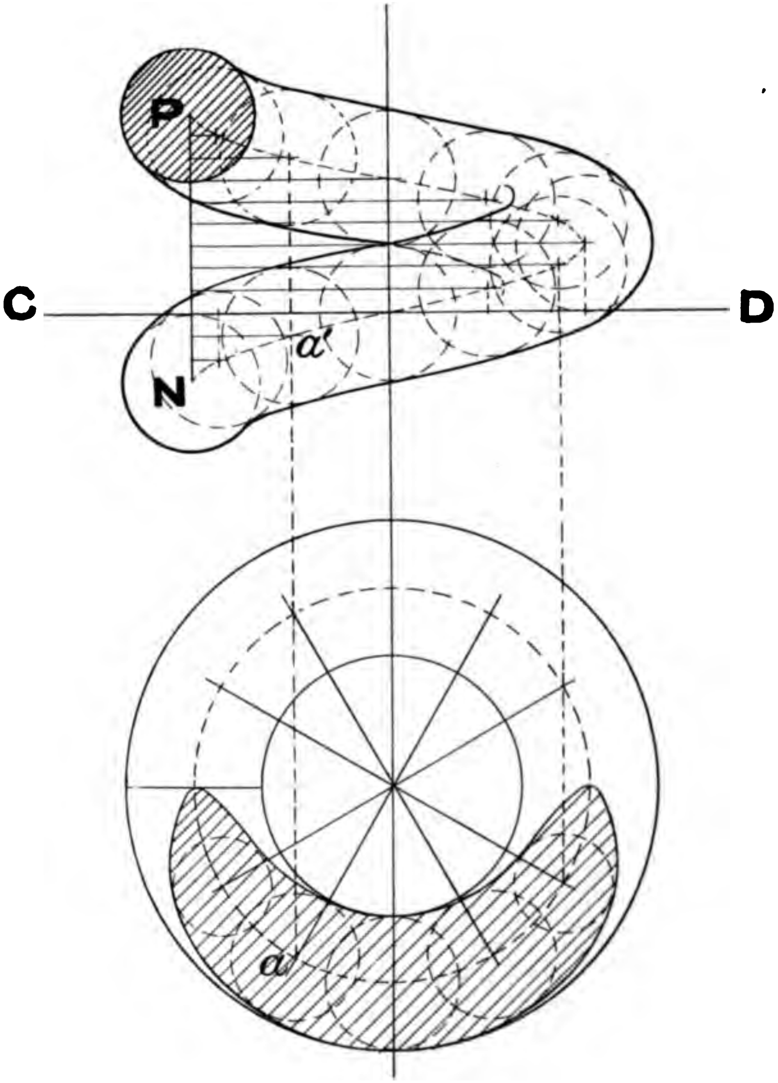
For the embankment on the near side, take *xy* perpendicular to *cd* through *c* and *d*, and sloping downwards at an angle of 45°, and determine level 20 ft. on each, and their plans, being joined, will give the level 20. The levels on the right and left side are found in exactly similar manner, and completed as before.

No. 59, 1892. **A sphere N moves uniformly round a vertical axis $a'b'$, and traces a helical surface as it rises into position P. Draw the elevation of this surface, and make a complete horizontal section of it on the line CD.**

First of all, obtain the elevation and plan of the spiral path of the centre of the sphere. This is done in the same

---

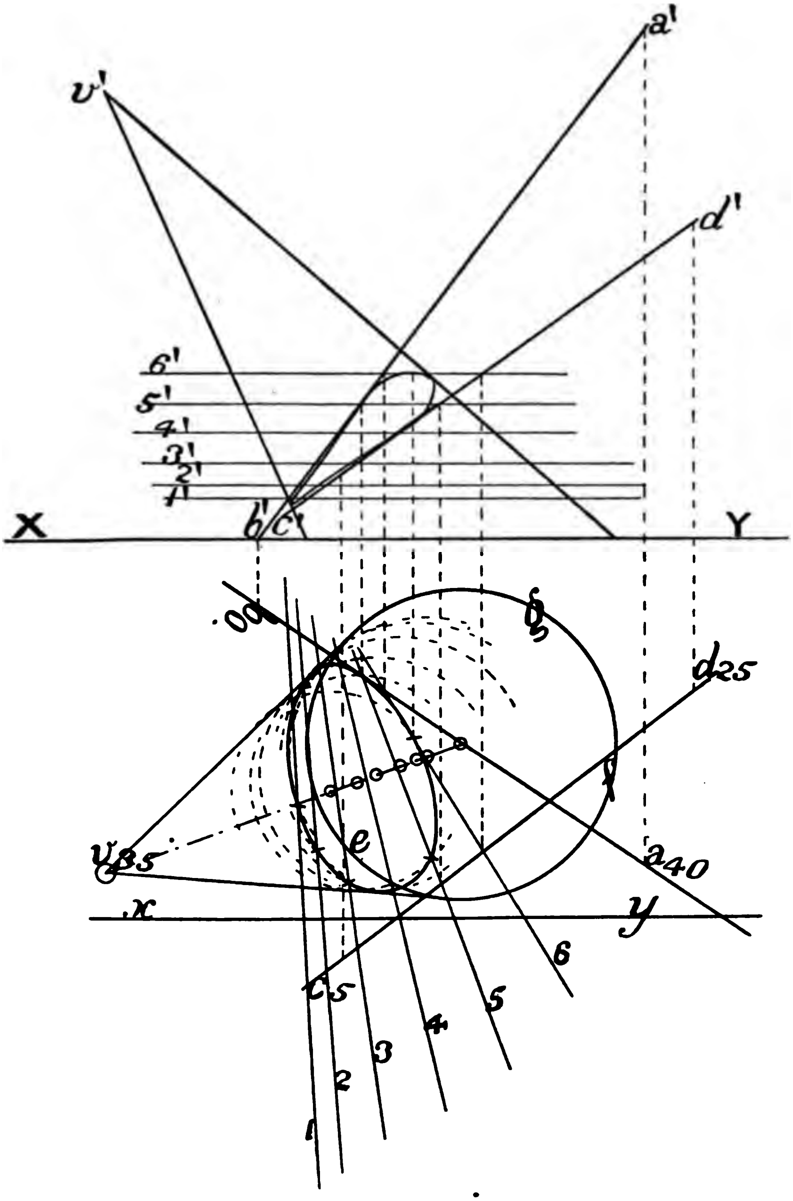
* Eagle's "Descriptive Geometry," Prob. 44.


P
C
D
a'
N
a

manner as the ordinary screw thread, whose pitch is NP. The plan, of course, will be a circle. With the several points thus found on the elevation of the spiral as centres describe circles; then draw curves that will just enclose all the circles as shown. Next cut this helix by a horizontal plane whose vertical trace is CD. This plane will cut the spheres in horizontal circles, which are shown dotted in plan, and the curve which encloses all these circles is the outline of the section. The section is shown shaded.

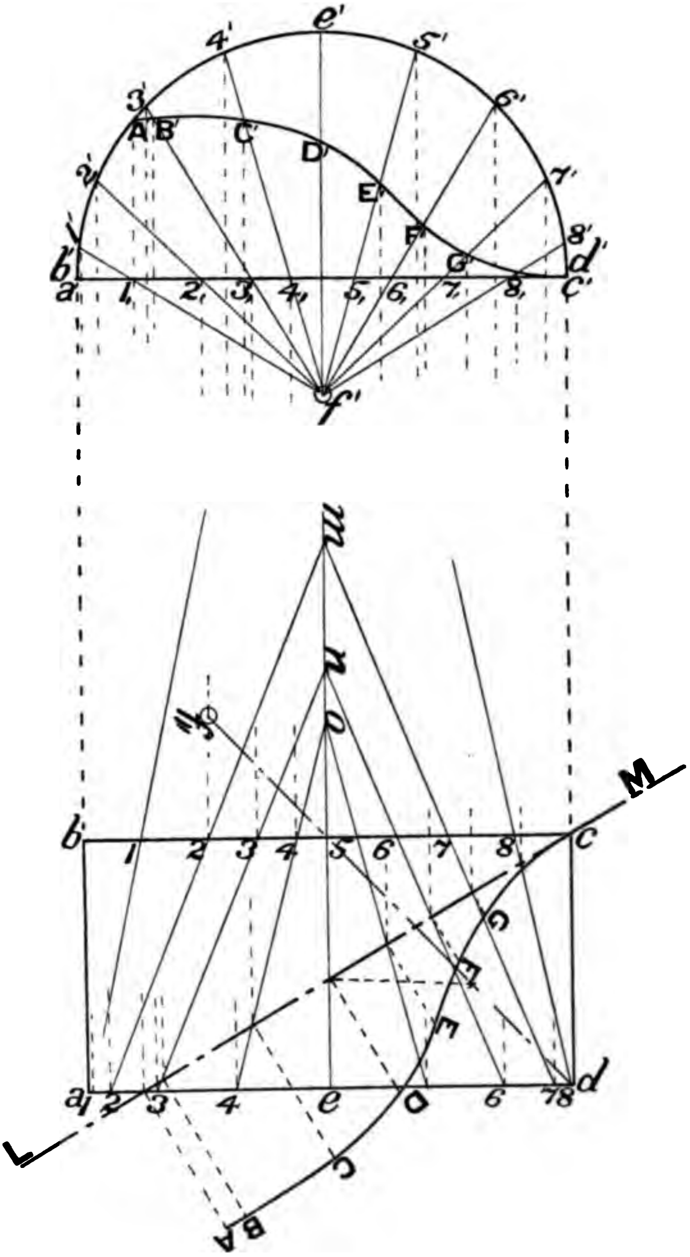
No. 51, 1888. **A twisted surface is traced by a horizontal line meeting the given lines *ab*, *cd*; *efg* is the horizontal trace of the conical surface whose vertex is *v*. Draw the plan of the intersection of the two surfaces, and also an elevation on *xy* (unit 0·1 in.).**

The lines *a*40, *b*0, *c*5, *d*25, the base *efg*, and *v*35 are given, also *xy*. It will be noticed that the base of the cone is a circle, therefore all horizontal sections will be circles. Make an elevation of the cone and the lines on XY. The ground-line XY is taken parallel to *xy*, but far enough away not to fall on the plan. Take next a series of horizontal planes as 1′2′3′4′5′6′, cutting the cone in circles, whose plans are shown in dotted lines; join the points where these planes cut the lines shown in plan at 1, 2, 3, 4, 5, 6, the lines cutting the circles in plan, *e.g.* line 1 cuts circle 1 in two points, line 2 cuts circle 2 in two points, and so on for each line. The curve drawn freehand through these will be the curve of intersection; each of these points projected up to their proper cutting planes will give the elevation of the intersection. The solution is obvious from the drawing, and needs no further description.


a'
v'
d'
6'
5'
4'
3'
2'
1'
X
b'
c'
Y
o
δ
d25
f
v35
e
a40
x
y
c5
6
5
4
3
2
1

No. 50, 1889. **A square (*abcd* in plan) is to be covered in by a surface generated by a straight line, which always intersects the semicircle *aed*, the line *cb*, and a line the elevation of which is point *f*. Draw the plan and elevation of the generator in at least eight positions, and determine the section of the surface by the vertical plane through LM.**

Draw the generators in elevation as shown at 1*f*, 2*f* . . . 8*f*; then, as the generator is to touch the line *cb* and the line f_1 drop projectors from their crossing-points (in elevation) with semicircle, and the line $c'b'$ to their plans respectively, these points being joined and produced to meet line *f* in plan at *m*, *n*, *o*, etc., give the surface required. The section is obtained easily by projecting the crossing-points (in plan) with generators to their elevations, as at A′, B′, C′, . . . G′; then the curve drawn through these points will be the section on the line LM. Its true shape is shown on the plan as ABC . . . G; the generator is rabated to the ground at *df*, and the height at which the plane LM cuts it is discovered.


e'
4'
5'
3'
6'
A
B
C'
D'
E'
F'
G'
2'
7'
1'
8'
b'
d'
a
1
2
3
4
5
6
7
8
c'
f'
m
n
o
f
b
c
M
L
a
d
e
D
C
B
A
E
F
G

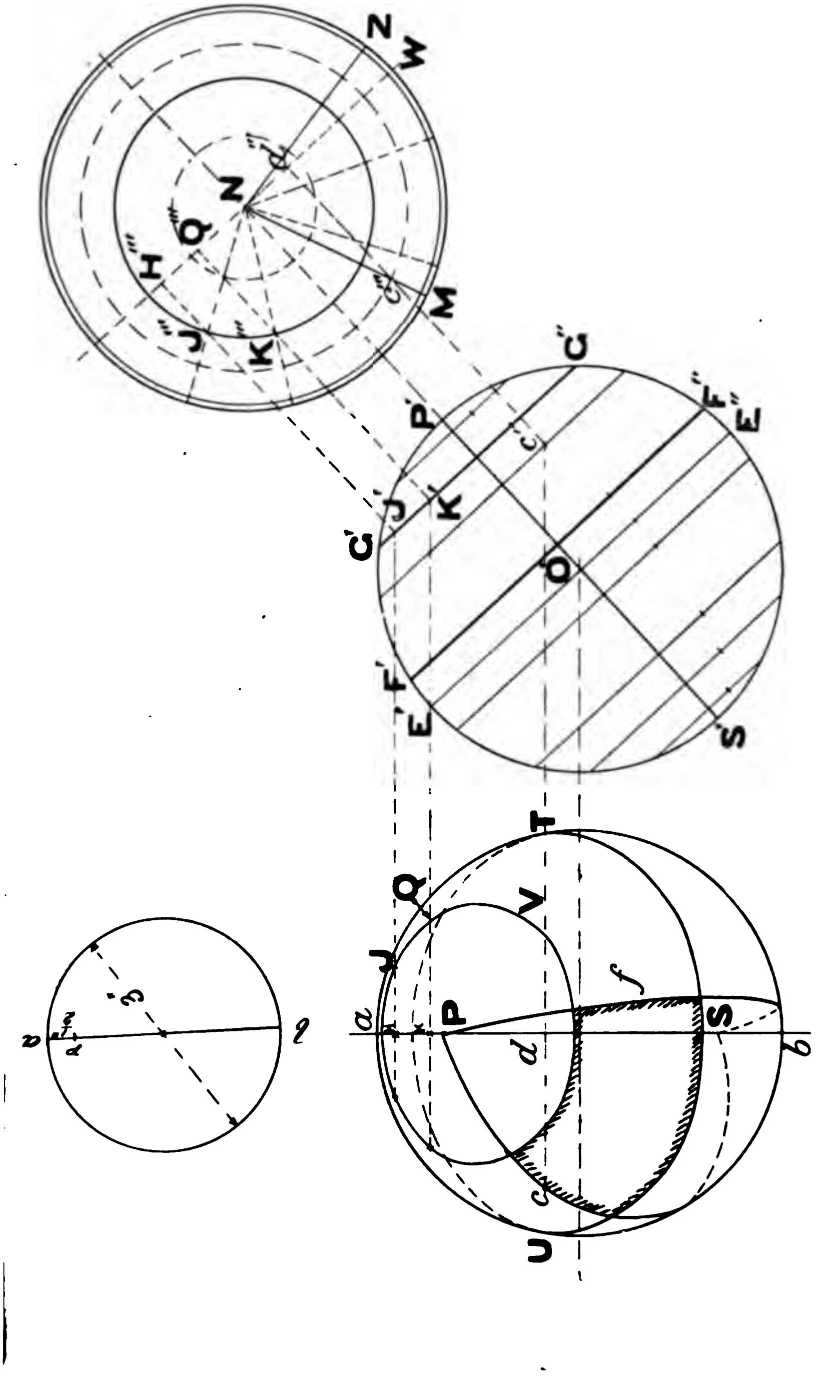
No. 61, 1890. **Draw the figure in accordance with the given dimensions. The circle represents a globe in elevation, P being the North Pole, and *ab* longitude 0°. Trace on the elevation the circles of 10° and 50° north latitude, and obtain the elevation of the area lying between these latitudes and between 10° east and 70° west longitude.**

Reproduce the elevation of the globe to the enlarged scale, with north pole at P and south pole at S. Take a side elevation of the globe looking from left to right. The poles are at P′ and S′.

Mark off E′OF′ = 10° towards the north pole from the equator. Set off E′O′G′ in a similar manner equal to 50°. Draw F′F″ and G′G″ perpendicular to P′S′. These lines are the side elevations of the two circles of latitude. In S′P′ produced take N, and with it as centre describe circles with F′F″ and G′G″ diameter. This figure will be an axial view of the circles of latitude.

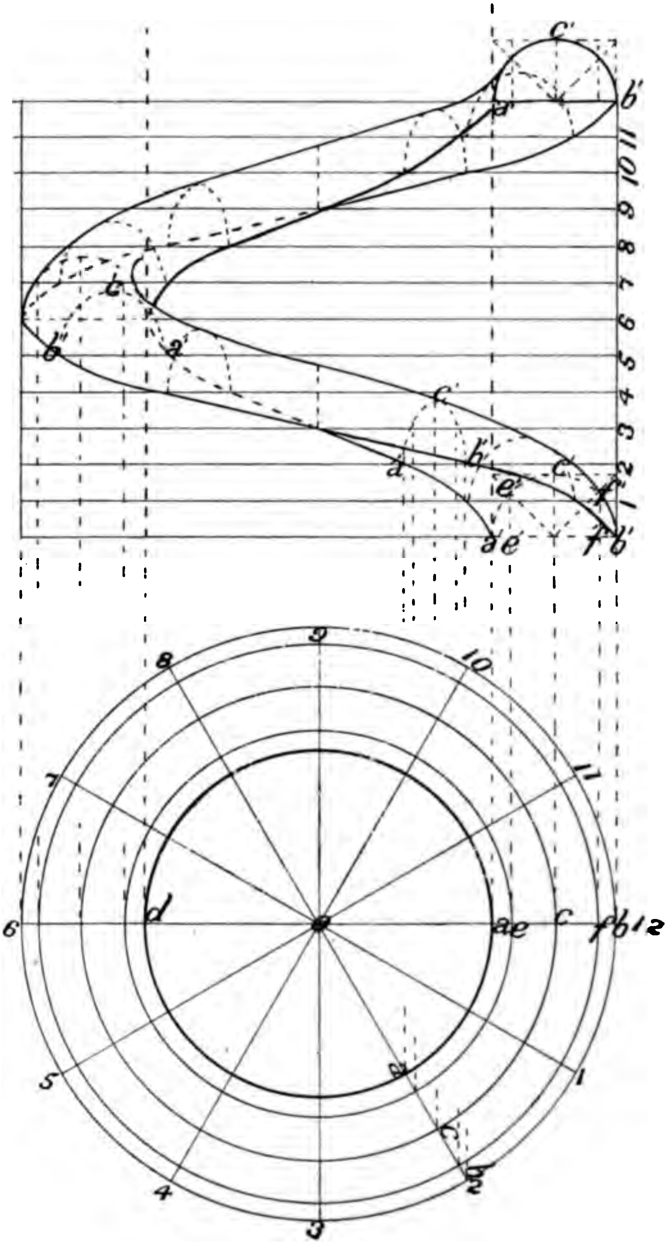
Now obtain the elevation of the circles G′G″ and F′F″. Divide up the circle into say twelve equal parts in J‴, K‴, etc. Project to the side elevation, and thence to the front elevation, measuring off JH = J‴H‴, etc. In this way the ellipses JQV and TRU are obtained.

To get the circles of longitude, set off the angle WNZ = 10°, and the angle WNM = 70°. The point *c*‴ is in side elevation at *c*′, and in front elevation at *c*, where *cd* = *c*‴*d*‴. In this way the ellipses P*c*S and P*f*S are obtained. The shaded portion is the elevation of the required area.


Z
W
N
d''
Q'''
H'''
J'''
K'''
c'''
M
C''
F''
E''
P'
c''
J'
K'
C'
O'
F'
E'
S'
T
Q
V
J
P
a
S
b
f
d
c
U
a
b
c
d
e
f

No. 49, 1889. **A screw surface is generated by the revolution of the vertical semicircle *abc* round the vertical circular cylinder, the diameter of which is *ad*. It rises from the position $a'c'b'$ to $a_1'c_1'b_1'$ in one revolution. Draw the elevation of the surface.**

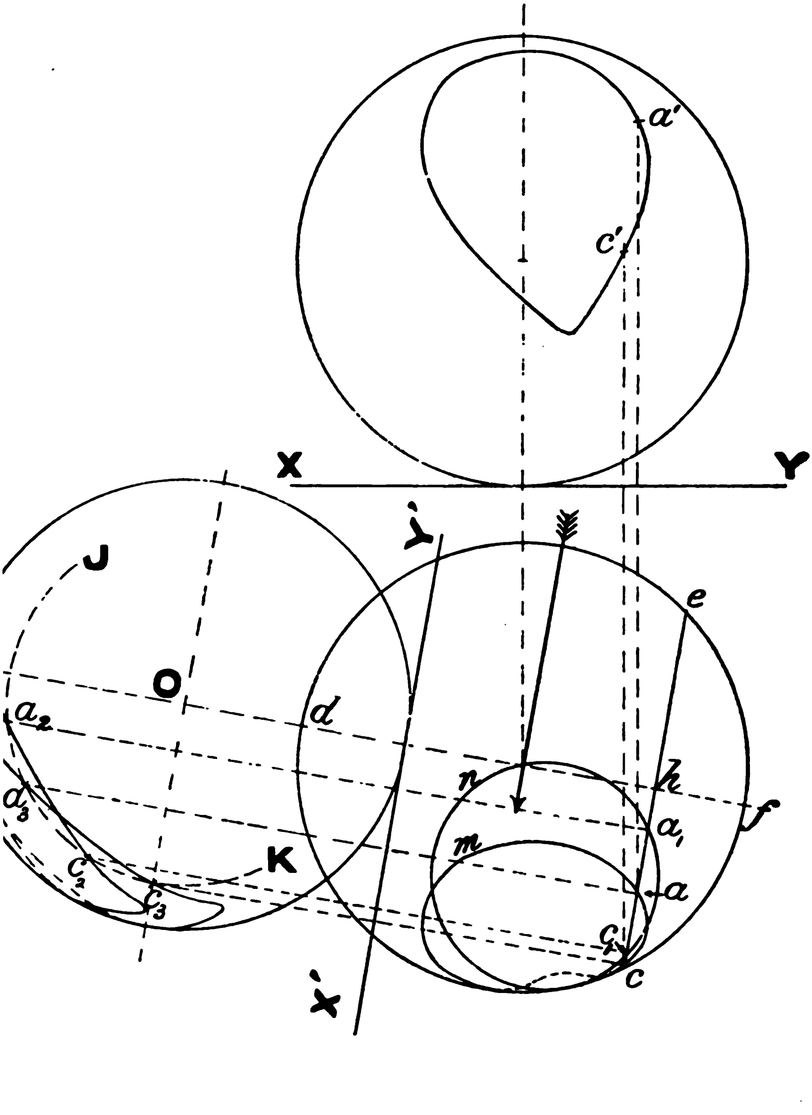
The circular base diameter is *ad*, the plan of semicircle is *acb*, and the elevation of cylinder and semicircle $a'c'b'$ are given; the generator evidently is intended to move round the cylinder with its plane passing through the axis. Commence by enclosing the generator semicircle $a'b'c'$ with a rectangle; join its corners to its centre, cutting the generator in points *e* and *f*. As each of these points move round the cylinder, they will have their plans in a circle, with *o*1 as centre, radii *oe, oc, of, ob.* Describe these circles as shown, and divide the outer circle into any number of equal parts, twelve here shown. Divide also the height through which the generator rises into the same number; then, as the generator moves to point 1, it will also have risen to line 1; as it moves to point 2, it will also rise to line 2, etc. Project then the points *a, e, c, f, b* in plan to their levels in elevation, giving eclipse $a_2'e_2'b_2'$, and so on for every position of the generator. The curve tangential to these ellipses will be the outline of the surface generated.


c'
b'
a'
11
10
9
8
7
c
6
b''
a
5
4
c'
3
d
h'
2
e'
c
f
1
d
e
f
b'
9
8
10
7
11
6
d
ae
c
f
b
12
5
1
d
c
b
4
3

No. 60, 1890. **Draw full-size two circles touching each other as shown, *efc* (diameter $2\frac{7}{8}$ in.) and na_1c_1, the latter being half the diameter of the former. The smaller circle represents the plan of a curve lying on the surface of the sphere, of which the larger circle is the plan. The vertical axis of the sphere is tilted in the direction of the arrow till it makes 70° with the ground. Make an elevation of the sphere on XY as ground-line.**

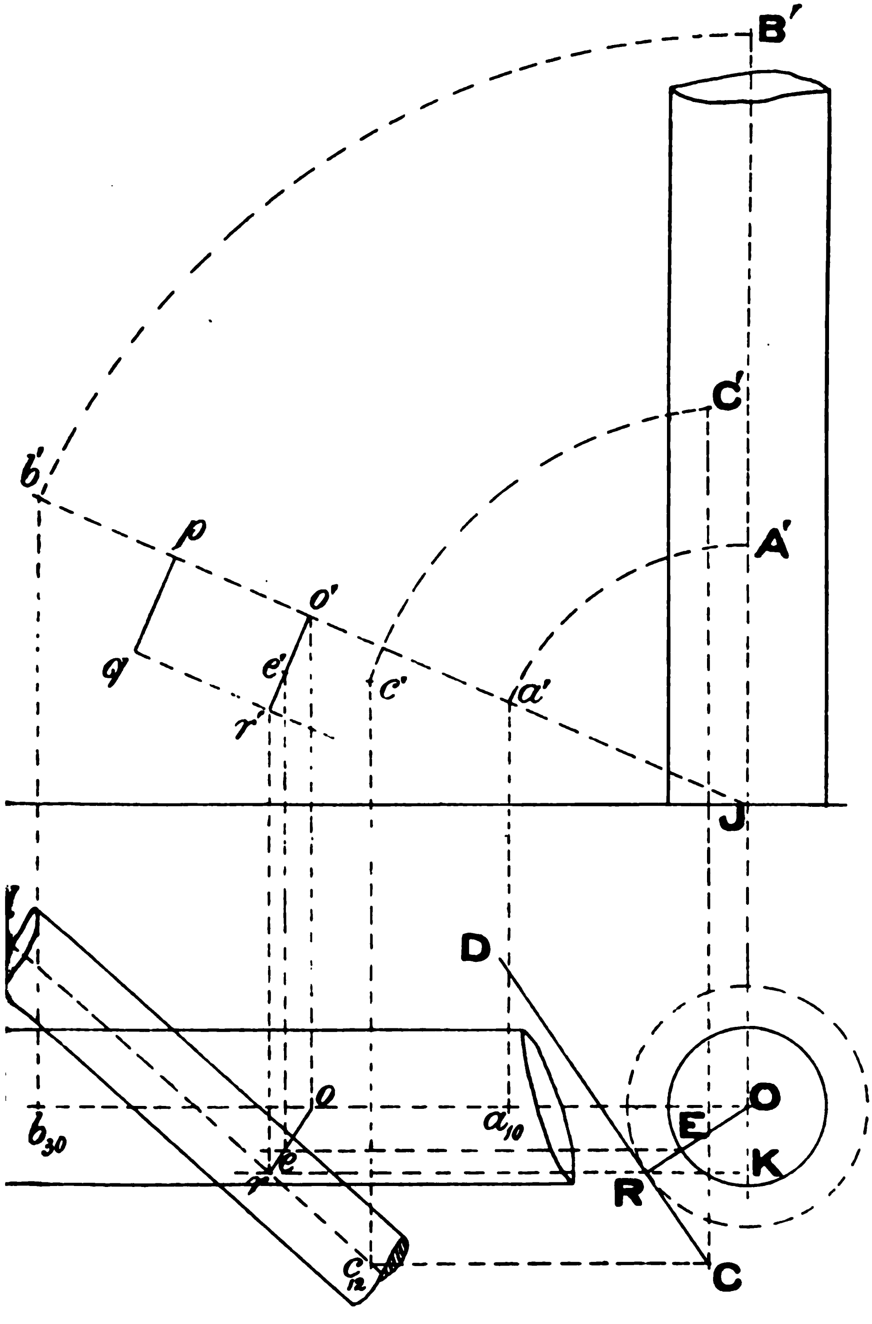
Cut the sphere by a series of vertical planes parallel to the arrow, one of which is shown at *ehc*. The outline of the section is the circle JK. The small circle na_1c_1 is cut at a_1 and c_1, whose positions in elevation are a_2 and c_2 Taking other sections in the same way, the curve a_2c_2 is obtained.

Now turn the sphere about an axis perpendicular to the paper at O, through 20°. The curve a_2c_2 will be moved to the position a_3c_3 in elevation and *ac* in plan. The elevations of *a* and *c* are *a*′ and *c*′. In this way the closed curve *a*′*c*′ is obtained. The heights of *a*′, *c*′, etc., above XY are the same as the heights of a_3 and c_3 above X′Y′.


a'
c'
X
Y
J
y
e
O
d
a₂
h
n
f
a₁
d₃
m
K
c₂
a
c₃
c₁
c
x

No. 59, 1890. **Two right cylinders (shown in plan) touch each other; the axis of one cylinder and a point in the axis of the other are given. Determine and figure the point of contact.**

Make an elevation, $a'b'$, of the given axis ab on a ground-line parallel to the plan of the axis. Produce the elevation to cut the ground-line in J. Rotate the cylinder (whose axis is AB) about the point J until it is in a vertical position at A′B′. The elevation of the point c_{12} is at c'. Rotate this elevation through the same angle about J to C′. The plan C of C′ is at the intersection of the vertical through C′ and the horizontal through c_{12}. The plan of the cylinder is shown by the full circle (centre O). The axis CD of the other cylinder will in plan touch a circle (shown dotted) whose radius equals the sum of the radii of the cylinders. Join O to the point of contact R, cutting the full circle in E. This point E is the plan of the point of contact of cylinders in this new position. Now rotate the whole back to its original position, carrying the point E with it to e. Thus: draw the horizontal Ee; then the plan of the point must lie in this line. The end R of the line OR lies in the axis of the second cylinder; hence project R horizontally until the projection cuts dc_{12} in r. Then r is the plan of the end of the line OR. Its elevation will lie vertically above it somewhere in rr'. The distance of the elevation of R from the elevation of the axis of the first cylinder is RK. Hence set off pq perpendicular to $b'a'$ and equal to RK. Through q draw qr' parallel to $b'a'$, cutting rr' in r. This is the elevation of the end of OR. The elevation of OR will be perpendicular to $a'b'$ in any position; therefore through r' draw $r'o'$ perpendicular to $a'b'$, cutting $a'b'$ in o'. Find the plan of o' by projection, and draw or cutting Ee in e. This is the plan of point of contact. Its elevation e' is obtained by projection. The second cylinder passes under the first. A second position is over the first, and the point of contact will be above the axis.


B′
C′
A′
b′
p
o′
q
e′
c′
a′
r′
J
D
o
a_{10}
b_{30}
e
r
E
O
K
R
C
c_{12}

No. 54, 1896. **$G'v'f'$ represents the elevation of a cone, its axis, $v'o'$, parallel to the vertical plane, its generating lines inclined at 40° to the axis. Determine a section of the cone which will give an ellipse whose major axis is $2\frac{1}{4}$ in., and focal distance $1\frac{3}{4}$ in. in length.**

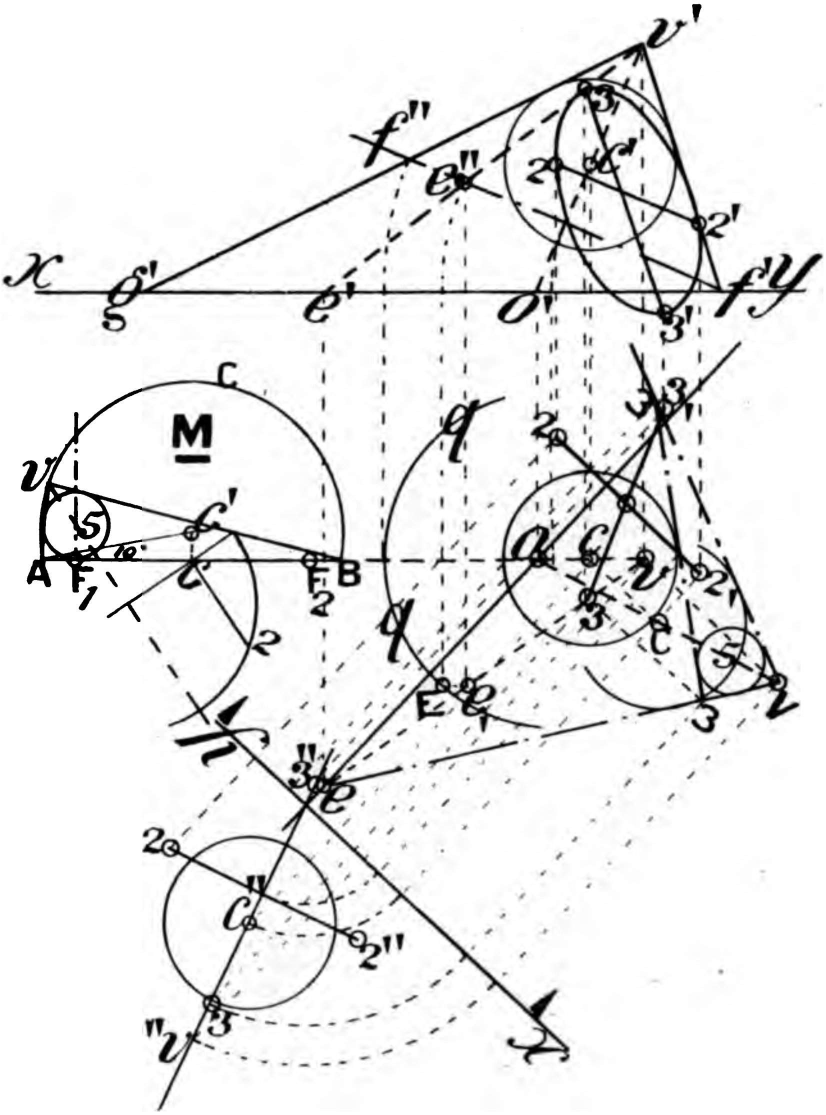
The upper extremity of the major axis of the ellipse is to be on the generating line VE ($v'e'$ in elevation). Draw the elevation of the ellipse.

The generating lines $v'g'$, $v'f'$, the axis $v'o'$, and the generator $v'e'$ are given; the semivertical angle at the vertex is 40°.

Now, we may commence by first determining the plan of the cone (the elliptic trace of its base in the horizontal plane need not be drawn). In the cone inscribe a sphere with centre on the axis $v'o'$, as at c. The plan will be on a line parallel to the V.P. at c, and also the plan of the vertex v. Next find the plan of the generator $v'e'$. Draw $f'f''$ (see elevation on xy) perpendicular to the axis $v'o'$, cutting the generator $v'e'$ in e''; now rotate this plane to the H.P., carrying the point e'' with it to E, which is evidently on the circle which is the section of the cone through $f'f''$; and rotate it back to its original position, when E comes to e. Then ve is the plan of the generator $v'e'$.

Now determine the plane containing the generator $v'e'$, ve and the axis of the cone, in the usual way, and make an elevation of the whole on a ground-line x_1y_1 perpendicular to the H.T. of that plane.

Rotate the plane to the H.P., carrying $v''c''$ to VC. With C as centre, draw the circle representing the sphere, and also the generating lines Ve, V3; the point 3 is that in which ve touches the sphere.


v'
f''
e''
3,
c'
2
2'
X
g'
e'
d'
3'
f'Y
g
C
M
q
2
3 3,
v
5
c'
A
F
1
10
c
F
2
B
a
c
v
2,
3
2
q
C
5
E
e
1
3
2
t
3''
e
2
c''
2''
3
''v
t

This is evidently the plane of central section of the cone, containing the generator $v'e'$, ve, and also the axis, rabatted to the horizontal plane, and its end elevation is shown on x_1y_1. The plan of section required must be perpendicular to this plane, in such a way that the section cut by it is an ellipse whose major axis is 2¼ in. and its focal distance 1¾ in. in length as given.

Now draw a line, AB (fig. M) 2¼ in. long, and mark F_1F_2 equal to 1¾ in., to represent the required focal distance, and on AB describe a segment of a circle to contain an angle of 80° (*i.e.* the vertex angle of the cone) in the usual way. At F_1 erect a perpendicular as shown, and determine the elevation of a sphere, having its centre on this perpendicular, in contact with F_1, so that the generators VA, VB are in contact with it, and having its vertex in the circumference of the segment ACB, as shown. Then the △ AvB evidently represents a part of the cone whose vertical angle is 80°, cut off by a plane AB; and the elliptic section (major axis AB) of the cone * will also have its major axis = AB, its focal distance = F_1F_2, and its minor axis = C2.

Now transfer vA, vB (fig. M) to the rabated section at V3, V3, with its inscribed sphere centre 5. Then 3 3V is a facsimile of ABv (fig. M). Bisect 3 3, and project this point and the major axis to the ground-line x_1y_1, and rotate the plane containing them back to its original position; then 3 3 will come to 3 3″. At the centre draw 2 2″ perpendicular to the plane, and equal to C2 (fig. M). Then 3 3″, 2 2″, in the elevation on x_1y_1, are the major and minor axes respectively of the section required to be cut off the cone. Project these to their plans as shown at $3\,3_1$, $2\,2_1$, find their elevation, and complete the ellipse. Then the ellipse shown in the elevation axes $2\,2'$, $3\,3'$, will be the required section satisfying the given conditions (a part of the lower lip comes below *xy*).

---

* It is a well-known theory that, if a cone be cut by a plane, the sphere inscribed in the cone tangential to the cutting plane will first have its contact point with the plane at the focus of the resulting curve, whether ellipse, hyperbola, or parabola, and the point where the plane cuts the outside generator of the cone will be the vertex of the curve, and the line in which the plane of the circle of contact of cone with sphere intersects the cutting plane will be the directrix of the curve (see Wilson's "Solid Geometry," p. 93). Hence the above reasoning has been used to determine the inscribed sphere centre 5, fig. M.

No. 48, 1888. **Draw the plan of a right circular cone, diameter of base 3 in., height 3·5 in., touching the given planes A and B, and so placed as to be above the plane A, and below plane B (unit = 0·1 in.).**

The scales of slope marked A and B are given; construct a cone to the given data, and in it inscribe a sphere (Fig. 1). Now determine the intersection PV of the planes A and B; and then determine the intersection of a plane C (whose distance above plane A is *r*, the radius of the inscribed sphere), and a plane D at the distance *r* below the plane B. This intersection is QS; then QS is a distance *r* away from both planes. Select any point S on this line as centre, and draw the plan of the sphere. Now determine a point V on the intersection of planes A and B whose distance from S = S to the vertex V (Fig. 1). To do this, rabat the plane containing both intersections to the ground, as at QS_1 and PV_1, when the true length S_1V_1 is readily seen. V_1 rabatted back to its plane will come to V: then SV is part plan of the axis of required cone. Make an elevation of the sphere and point V on X_1Y_1 parallel to the plan of axis of cone. Heights of S″ and V″ are easily obtained from the end view of plane containing the intersections, as at 1S, 2V on *xy*. Complete the elevation of cone, and determine its plan as shown.

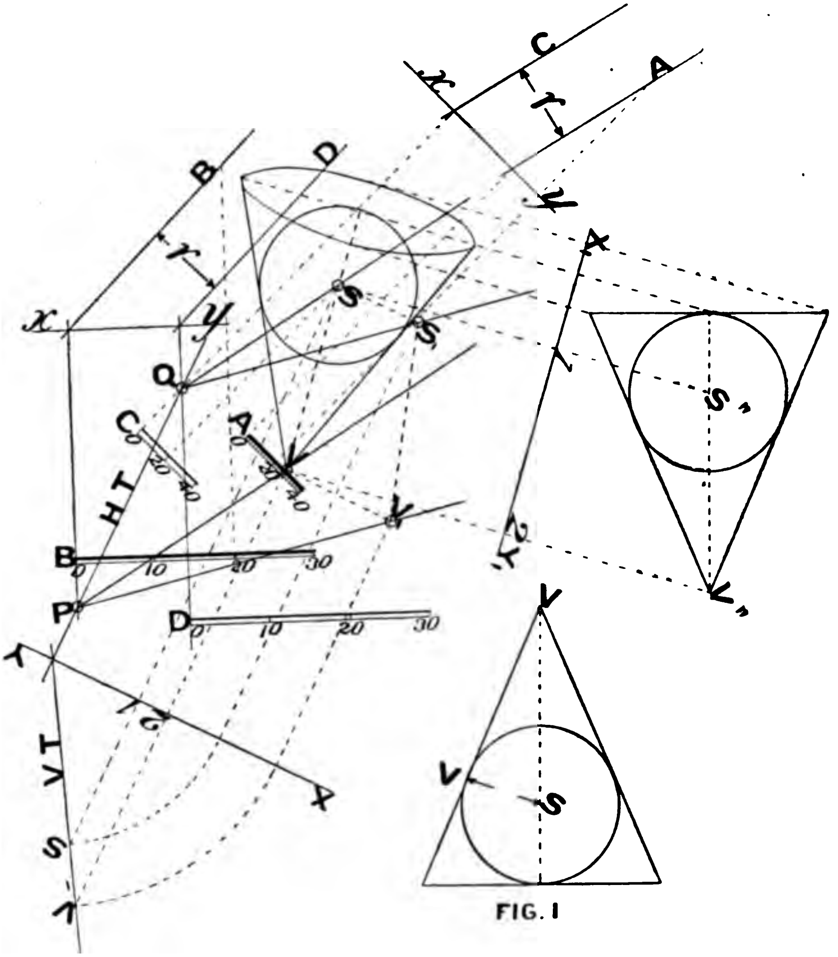
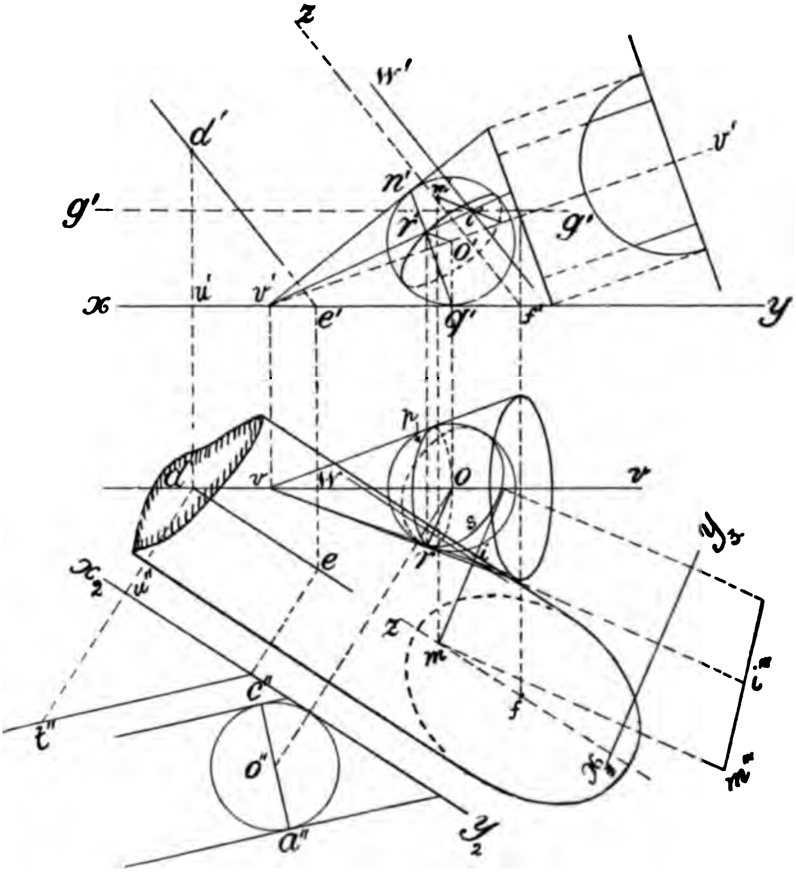

C
A
x
r
B
D
r
x
y
S
S
Q
C
0
20
40
A
0
30
40
V
HT
V
B
0
10
20
30
P
D
0
10
20
30
Y
r₂
X
VT
S
V
X
r
S"
Y
2
V"
V
V
S

FIG. I

No. 55, 1893. **A cone is shown in elevation whose axis is *v'v'*, parallel to the vertical plane. The cone lies on the ground. A cylinder of 1 in. radius, with its axis parallel to the given line *ded'e'*, *touches* the cone at a point in the same horizontal plane as *g'g'*. Draw the plan of the surfaces and the projections of their point of contact, showing the trace of the cylinder in the horizontal plane, and the projections of its generator in contact with the cone.**

The axis parallel to the vertical plane is shown in plan at *vv*. A sphere is inscribed in the cone. The circle of contact in elevation is the line *n'q'*, and in plan the ellipse *p*. Obtain a new elevation of the sphere on a ground-line x_2y_2 parallel to the plan *de* of the given line. It will be noted that *e* is the horizontal trace of *ded'e'*. Set up *u''t''* equal to *u'd'*; then *t''c''u''* is the true inclination of DE to the ground. Draw the tangent to the sphere at *a''* parallel to *t''c''*, and also the tangent on the other side of the diameter. If a line is to remain parallel to itself and be moved round the sphere in contact with it, the line of contact is the circle of which *o''a''* is the radius. The plan of this circle is at *s*, and the elevation is the ellipse shown in elevation on *xy*. The line through a point of intersection of the circles of contact, parallel to the given line DE is a possible generator of a cylinder touching the cone. Draw the generator of the cone through this point *rr'*. This generator is the locus of all points of contact between cylinder and cone. The point *ii'*, the intersection of the horizontal *g'g'* with the locus, is the point of contact required. Join *o'r'*, and through *ii'* draw *im*, *i'm'* parallel to *or*, *o'r'*. The axis of the cylinder will pass through a point in this common normal. Take a new elevation *m'''i'''* of this common normal on the ground-line x_3y_3 parallel to its plan, making *m'''i'''* equal to 1 in. The point *m* is thus found, and consequently *m'*. The axis of the cylinder is *zfz'f'*, and the generator in contact *wiw'i'*, both being drawn parallel to *de*, *d'e'*. Then draw the plan of the cylinder having a diameter of 2 in. The trace of the cylinder in the horizontal plane may be obtained by getting a new elevation on a ground-line parallel to *ab*. This elevation is not shown.



Another method of solution is here added by which fewer ellipses are necessary.

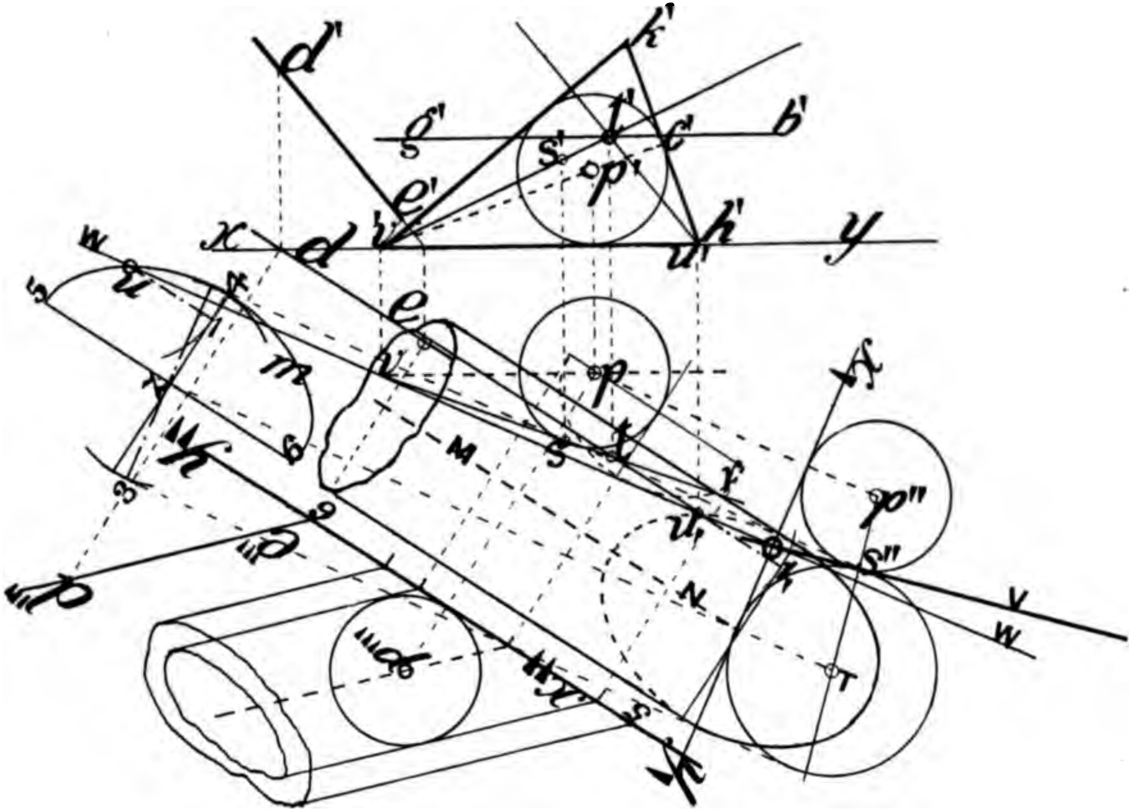
Thus, determine a plane tangential to the cone and parallel to line *ded′e′*, shown by its traces WW and VO, which is readily found by making a new elevation of the line *de* at $d'''e'''$ on $x_{11}y_{11}$ parallel to *de*, and also an elevation of the inscribed sphere centre p'''. Circumscribe this sphere by a cylinder whose axis is parallel to $d'''e'''$, having the elliptic trace in plan shown at *r*; then WW drawn through *v*, the vertex, and tangential to the ellipse, is the horizontal trace of the required plane.

The generator of contact is obtained from the end view on x_1y_1 perpendicular to WW, the sphere centre p'', and the vertical trace of the tangent plane is drawn. The point of contact with sphere is point S' and S. Project S to S' on *v′t′*, height above the ground the same as at S''. Then the generator *v′s′vs* is the line of contact of cylinder with cone in all possible positions; but as the cylinder is to be in contact with the cone at height *g′b′*, then the point *t′t* is the required point.

Now draw *t′u′tu* parallel to *d′e′de*, through this contact point. Then $t'u'tu_1$ is a generator of the required cylinder, which is contained by the tangent plane already found, and has its trace in point u_1u. It will be seen that *u* is a point in the elliptic base of the required cylinder.

To obtain the projection of the cylinder, draw $p''S''T$ perpendicular to OV, the vertical trace of the tangent plane, making $S''T$ equal to 1 in., and with point T as centre describe a circle 1 in. radius.

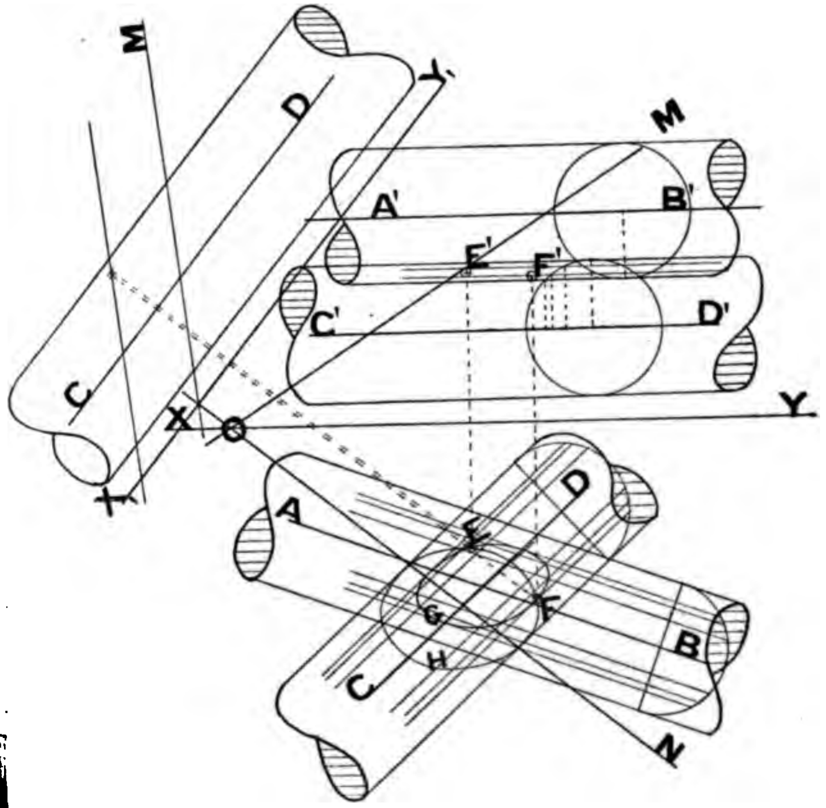
This circle is contained by a plane which is perpendicular to the tangent plane; consequently the plan of $S''T$ will be a line which is perpendicular to WW, and the plan of T will be a point in the axis of the required cylinder. Draw, therefore, the projector through T, and anywhere, as at *z*, draw a trial ellipse *um*, the minor axis ar to *de*, and the major axis parallel to *de*, and nd 5 6 respectively. The length 5 6 is readily


d'
k'
g'
t'
c'
b'
s'
p'
e'
v'
h'
x
d
u'
y
W
u
r
5
e
m
v
p
A
y''
6
s
t
M
3
f
6
e''
u,
p''
o
S''
d''
N
V
W
p'''
x''
5
T
h

found from the elevation of cylinder on $x_{11}y_{11}$ parallel to *de*; the point *u* corresponds on this ellipse to the point u_1 on the required elliptic trace of cylinder when in its required position. Conceive, then, this ellipse to move parallel to itself until *u* comes to u_1; *i.e.* to meet the horizontal trace of the generator of contact *tu, t'u.* Then the minor axis will be as far from u_1 as it was from *u* in the trial ellipse. The major and minor axes, of course, will be parallel and equal to the axes of the trial ellipse. Complete the ellipse, and draw the plan of the cylinder, MN being its axis. Its elevation is not drawn, but is left as an exercise to the student.

No. 45, 1888. **Determine the projections of a sphere (radius $\frac{3}{4}$ in.) touching the given plane MON, and also the two given lines A'B'AB, C'D'CD.**

Consider the lines A'B'AB, C'D'CD as axes of cylinders, radius $\frac{3}{4}$ in.; then the centre of a sphere touching both lines must be on the intersection of these two cylinders, at a point $\frac{3}{4}$ in. from the given plane MON. Draw the projections of these cylinders, and determine their intersection GEF. To do this, take a series of horizontal cutting planes (as both axes are horizontal) cutting the cylinders in generators parallel to the axes. The crossing-points of these generators will, on being joined, give the intersection of the cylinders. Now obtain a new elevation of the plane MON on X_1Y_1 perpendicular to NO, and also of one of the cylinders, such as the cylinder whose axis is C'D'; and determine the section of the cylinder made by a plane parallel to the given plane MON and $\frac{3}{4}$ in. away from it, shown in plan at HEF. These two intersections cross one another in points E and F; then E and F are two points fulfilling the given condition. Either of these projected to their proper heights, as E'F', will be the point required.


M
D
C
Y_1
X_1
X
O
Y
M
A'
B'
E'
F'
C'
D'
A
D
E
F
G
H
C
B
N

No. 61, 1891. **Explain what is meant by the principal axes of an oblique cone, the trace of which on the horizontal plane is an ellipse, and the plan of the vertex a point on the major axis.**

The principal axes of an oblique cone are the three lines drawn through the vertex, each of which is perpendicular to the plane of the other two, their lengths being such that a section of the cone made by a plane through the extremity of any one, and parallel to the plane of the other two, is a conic, the centre of which is on the first line, and the axis being parallel and equal to the other two respectively.*

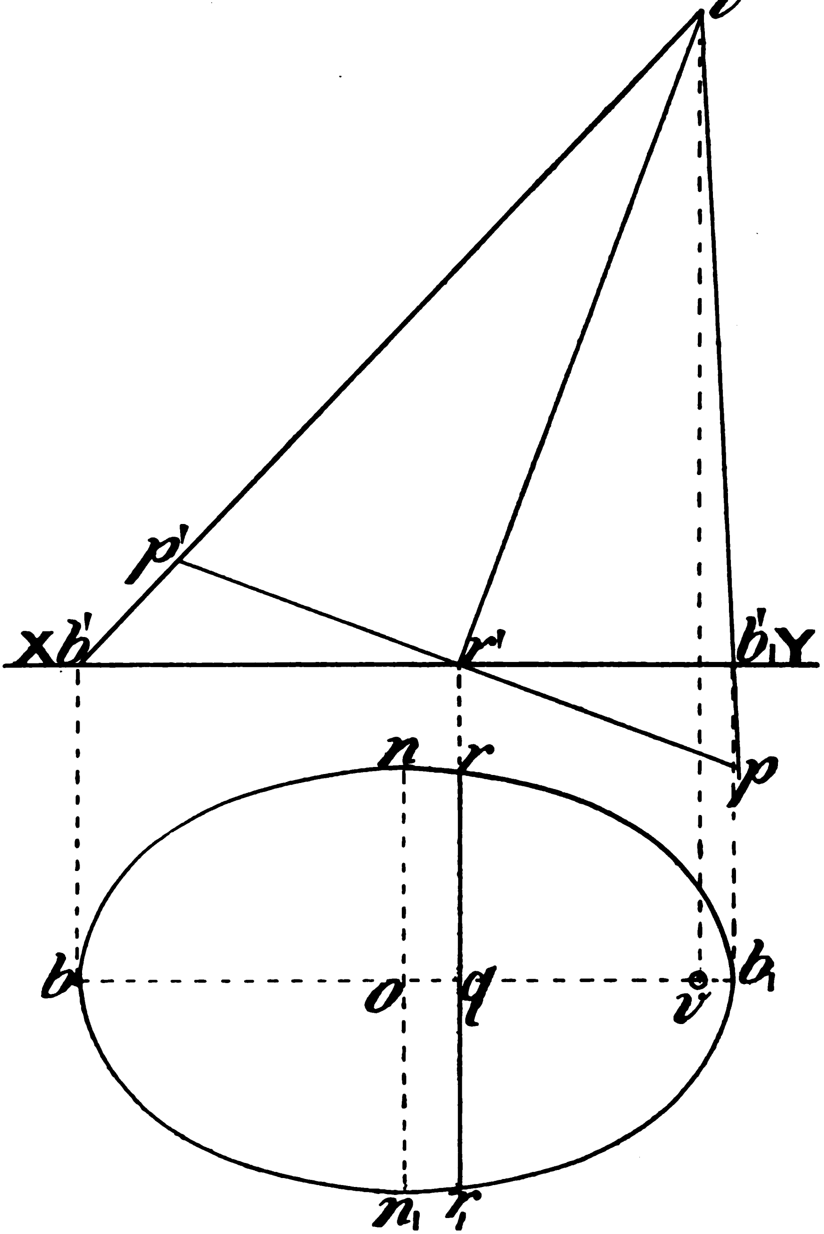
Let us consider the cone of which the ellipse (axes *bbnn*) is the plan; the vertex at *v*, anywhere on the major axis. Draw its elevation on a plane parallel to the major axis bb_1, which of course is the triangle $v'b'b_1'$.

The plane $b'v'b_1'$ is the principal plane of the cone, and two of three axes are contained by it, while the third is perpendicular to it.

Bisect the angle $b'v'b_1'$ by the line vr', meeting $b'b_1'$ in point r'; and draw $v'p'p'$ perpendicular to vr', meeting $b'v'$ in p', and $b_1'v'$ in p; then project r' to cut the ellipse in r and r_1. The lines $v'r'$, $p'r'p$ are the lengths of the two principal axes contained by the plane $b_1'b'v'$; the line rr_1 is the length of the third axis, which is perpendicular to the plane of the other two.

---

* For full discussion, see Eagle's "Descriptive Geometry," chap. xi.


v'
p'
X b'
r'
b'₁ Y
n
r
p
b
o
q
v
b₁
n₁
r₁

No. 56, 1891. **Circumscribe the given sphere (index of centre, 5) by a regular tetrahedron, one face touching the sphere in point P on its upper surface, and a second face passing through Q, which is level with centre of sphere (unit = 0·8 in.).**

The point pp', and c^5, the centre of sphere, are given. As one face of the tetrahedron is to touch the sphere in point P, it will therefore be contained by a plane tangential to the sphere at this point.

This plane may easily be found by taking XY parallel to cp, and making an elevation of the sphere with P on its upper surface, as at p', and VT, HT drawn perpendicular to $c'p'$, cp respectively will be the traces of the plane. (Note c, the centre of sphere, is taken at a higher level than that given, so that the solid may not come below XY; the level of c is immaterial for the working of the problem.)

Now draw $c'p'$ perpendicular to VT, and consider it as the axis of an imaginary cone circumscribing the sphere with a base angle equal to the dihedral angle between the two faces of a tetrahedron. This angle is determined by drawing any tetrahedron, and taking a vertical section through its axis and one of its inclined edges, as shown to a very small scale at Fig. 1, as 2,5,4; the angle marked ϕ is the dihedral angle between the two faces, H being the length of the axis.

Next make the angle $e'6'v'$ equal to ϕ, the line $6'v'$ touching the sphere, then $p'v'$ is the length of the axis of the tetrahedron and its inscribed cone $v'6'7'$, which envelops the given sphere.

The face which has to pass through the point q must evidently lie in the plane containing the vertex v, and the given point q', and must also be tangential to the cone $v'6'7'$, just obtained. Its intersection with the first plane, therefore, will be one edge of the base of the required tetrahedron. To find this intersection, project q to q' on a level with c', the centre of the sphere. Draw the line $v'q'$ intersecting the first plane in point $b'b$; now rabat this plane

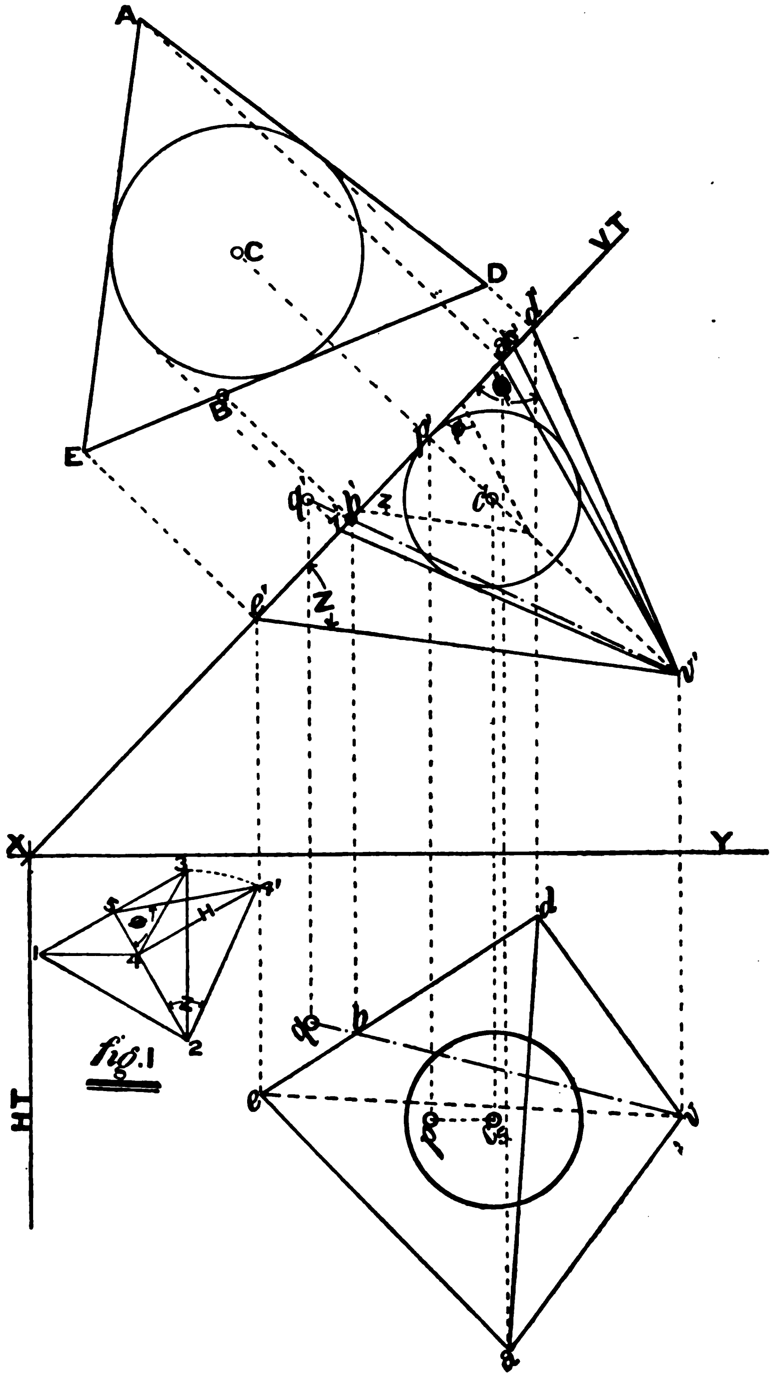
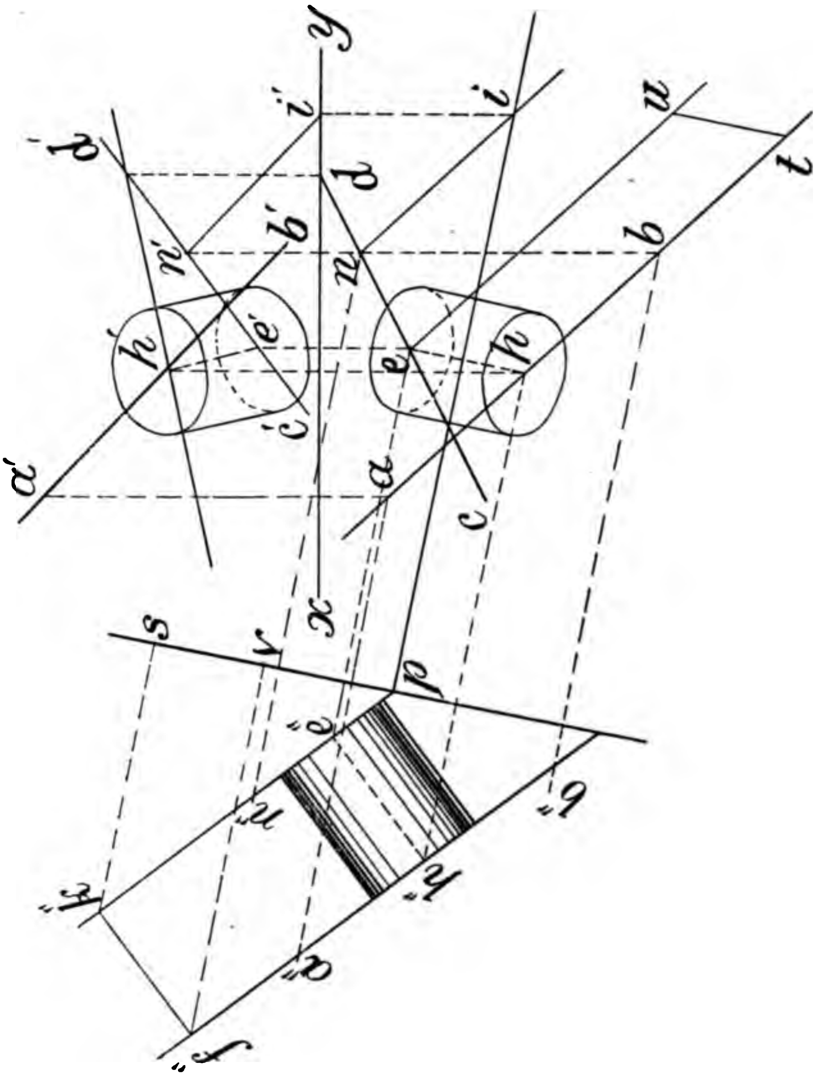


fig. 1

about its V.T. till it comes into the vertical plane, carrying the base of the cone centre *c* and point *b* with it to C and B. Of course C and B will be as far from VT as they are in front of XY. Then the line drawn through B tangential to the circular base of the cone is one edge of the solid, for it is the intersection of the two planes; the equilateral triangle EDA circumscribing the circular base is the face which is contained by that plane. Now rotate the plane back to its former position, carrying the triangular base EDA with it, which determines the plan *edav*, and the elevation *e'd'a'v'* of the required tetrahedron circumscribing the given sphere. "If necessary, *xy* may be now drawn 5 units below *c*, the centre of sphere, satisfying the conditions given."

No. 59, 1893. **Draw the projections of a right cylinder of 1 in. diameter whose axis is, in direction and length, the common perpendicular to the two given non-intersecting lines** *ab, a'b'* **and** *cd, c'd'*.

Draw the traces of a plane containing the line CD, and parallel to AB. Thus, in CD take any point N, and through N draw a line NI parallel to AB. The traces of NI and CD are points in the traces of the plane. Then *pi* is the horizontal trace, and *h'd'* the vertical trace. Take an end view of the plane on a new ground-line *ps*, *pk''* is that end view, and *a''b''* is the new elevation of AB. The perpendicular distance apart of *a''b''* and *pk''* is the length of the axis of the cylinder. Draw any such perpendicular, say *f'k''*. The length of its plan is *vs*. The axis of the cylinder being perpendicular to the plane, the projections of the axis must be perpendicular to the traces of the plane. Hence take any point *t* in *ab* and draw *tu* perpendicular to the trace *pi*, making *tu* = *vs*. Then through *u* draw *ue*, parallel to *ab*, cutting *cd* in *e*. The plan of the axis is *he*, where *he* is parallel to *u*, and the elevation of cylinder on *ps* is shown shaded. The ends of the cylinder are then obtained by projection.


y
i'
i
u
t
d'
d
b'
n'
n
b
h'
e'
e
h
c'
c
a'
a
x
s
v
p
e''
b''
n''
h''
a''
k''
f''

No. 56, 1898. **A tetrahedron ABCD stands on its base ABC on the horizontal plane, and its vertex D is at a height of 2 in. above the base.**

The face ADB is inclined at 65°
„ ADC „ 70°
„ BDC „ 61°
The edge DB „ 52°
„ DA „ 45°

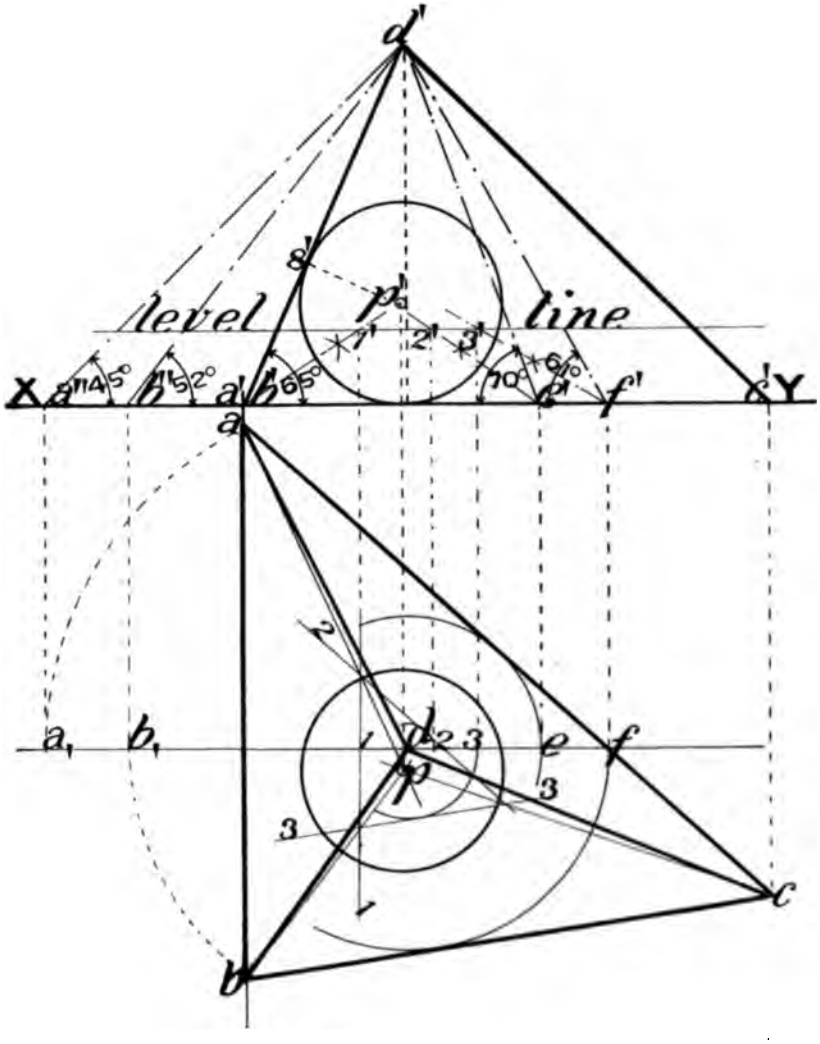
Draw the projections of the tetrahedron, so that the perpendicular from the vertex D on the plane of the base shall fall within the base, and draw the plan of the sphere inscribed in the tetrahedron.

As the plane of base is to be in the horizontal plane, we may immediately draw the plane of one face in its proper position, such as the plane of the face *adb* inclined at 65°, shown by its traces in elevation at $a'b'$, and in plan at *ab*.

Now select the point dd' which is to be the vertex of the solid, and is to be in the plane *adb* and 2 in. above the plane of base.

Next determine two lines, *db*, *da*, inclined 52° and 45° respectively, lying in this plane, thus: draw $d'a''$ inclined 45°, and $d'b''$ inclined 52°, having their plans at da_1 and db_1 respectively. Now with *d* as centre and radii *da*, *db* describe arcs cutting the horizontal trace of the plane in *a* and *b*; join *da*, *db*: then *da* and db_1 will be the required edges of the solid inclined 45° and 52°, and contained by the plane *adb*. Hence the triangle *adb* is one face of the required tetrahedron.

The horizontal trace of the other two faces will be the lines drawn from *a* and *b* tangential to two cones having their vertex at *d* and base angles of 70° and 61°, the generators of which are shown in elevation at $d'e'$, $d'f'$, and in plan by the arcs struck with centre *d* and radius *de* and *df*. Then *bc*, *ac*, drawn tangential to these arcs as shown, will be the other two sides meeting at *c*. Join *cd*, and the plan


d'
8'
p'
level
line
1'
2'
3'
X
a''
45°
b''
52°
a'
b'
65°
70°
60°
e'
f'
c'
Y
a
2
a₁
b₁
d
2
3
e
f
p
3
3
1
c
b

is complete. The elevation may be found in the usual way, hence the desired tetrahedron is shown in plan by the lines *a*, *b*, *c*, *d*, and in elevation by the lines *a′*, *b′*, *d′*, *c′*, and the perpendicular from the vertex *d* on the plane of base fall within the base.*

The centre of the inscribed sphere will be the intersection of the three planes which bisect the interior angles of the solid, which is readily obtained by drawing a level line as shown in elevation, and determining where the bisector of the angle *d′a′c′* cuts this level line as in point 1′1, the bisector of the base angle of cone *d′e′b′* cuts the level in point 2′, and the bisector of the angle *d′f′b′* cuts it in point 3. With *d* as centre, radii *d*2 and *d*3, describe arcs: then lines 2 2 and 3 3, drawn parallel to *ac* and *bc*, will be level lines on the bisecting planes of the dihedral angles of the solid; their crossing-points will be points in the intersection of the planes. The three intersections meet in point *pp′*; hence the circle struck with centre *p′p*, radius *p′*8, will be the plan and elevation of the sphere inscribed in the tetrahedron.

---

* Another tetrahedron might be obtained by drawing the line *ac* tangential to the base of cone *de*, passing to the left of perpendicular from *d*.

48, 17. **Draw a line through the point c to pass 1½ in. from each of the two given points a, b. State the conditions of possibility of this question.**

The given points are indexed $a0, b14, c30$. The required line will evidently be the generator of intersection of the two cones which envelop the two spheres, whose centres are a and b, and 1½ in. radius.

Commence, therefore, by drawing the axis ca, and with a as centre and radius 1½ in., draw the circle as shown. Now draw the enveloping cone in plan. Consider this axis as xy, and make an elevation of the cone; that is, make $cc''' = 30$ units, and draw the axis $c'''a$. a being in the horizontal plane, the cone in elevation will envelop the same sphere.

Determine the elliptic trace of the base of this cone on the horizontal plane as xx (only a part of it need be drawn), thus: bisect the distance xx, the crossing-points of the generators n_1o with cax in point 5; draw $5i$ perpendicular to $c'''a$; with i as centre, radius ii, describe the arc $i6$: then 5 6 is the semi-minor axis, and $5x$ is the semi-major axis. Now draw 5 6 at right angles with $5x$, and draw the part ellipse xx.

Now draw the enveloping cone to the sphere struck with centre b and radius of 1½ in., and make an elevation of this cone, taking cb as xy, thus: make $cc'' = 30$ units, $bb'' = 14$ units, drawn perpendicular to cb; then $c''b''$ will be the axis of the cone. Draw the elevation of the sphere centre b'', and draw its enveloping cone. The trace of this cone on the horizontal plane will be a parabola, which is easily determined by drawing the inscribed sphere centre o tangential to the horizontal plane; its contact point with the horizontal plane will be the focus, as at s; v, the point where the generator cuts the horizontal plane, will be the vertex; and the directrix ww is drawn perpendicular to the xy through the point 7, where the plane of circle of contact, 7 7, intersects the horizontal plane. The curve vu may be readily drawn by taking a series of ordinates perpendicular to *xy*, and taking their distance from the directrix as radius, and the focus *s* as centre, describe arcs cutting each ordinate respectively in points on the curve.* The curve *vu* intersects the elliptic curve in point *f*; then *f* is the horizontal trace of the required line. *c* is also a point on the line. Draw *cf*; then *cf* will be the generator of intersection of the two cones, which is 1½ in. from both the given points, for it is tangential to the spheres centres *a* and *b* respectively of 1½ in. radius, which fulfils the given conditions. Another line may be found to fulfil the same condition, by continuing the curves to cross on the under sides of the cones, which will be the other generator of intersection of the two cones.

Another method is also shown, on the same drawing, by which the same result is readily obtained without drawing either the elliptic or the parabolic curve of the two enveloping cones. Thus, find the traces of a plane to contain the lines *ca*, *cb*, as VOH; rotate this plane about its horizontal trace to the horizontal plane, carrying the lines *ca*, *cb* with it, when *c* will travel to C, and *b* to B, while *a* remains stationary, it being in the horizontal trace of the plane.

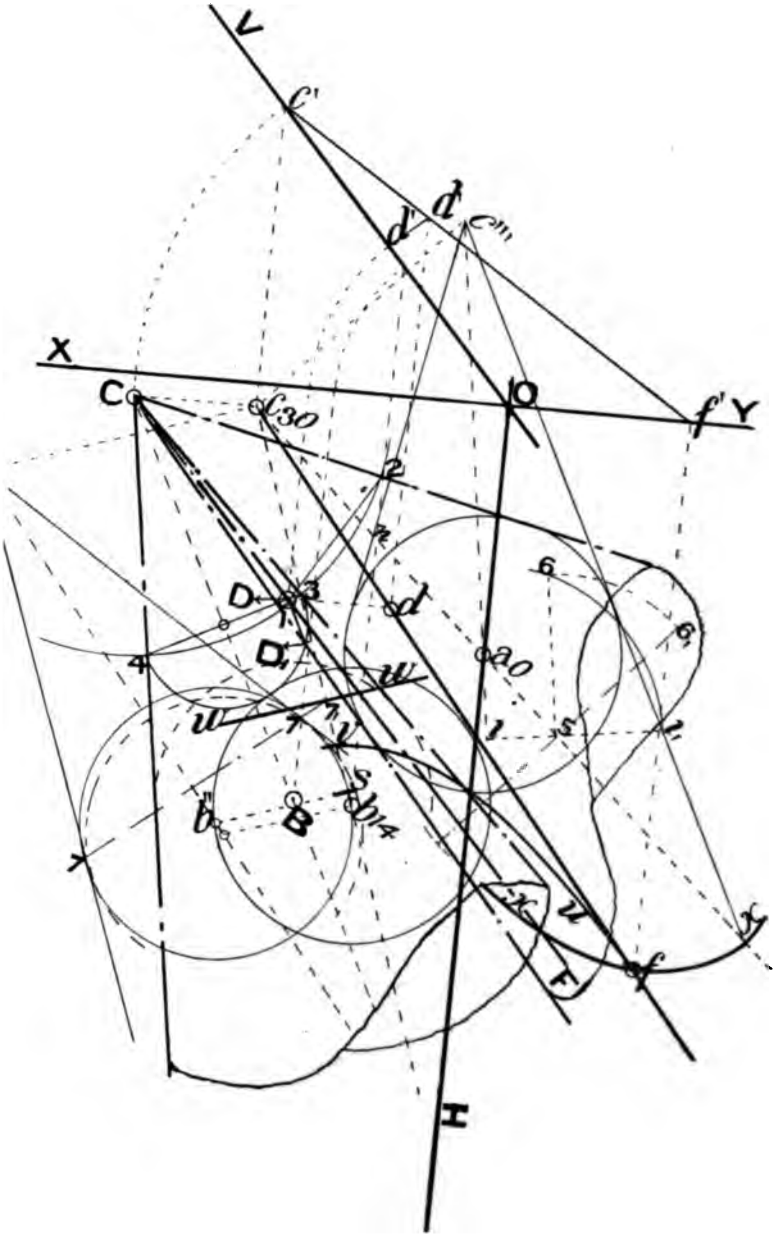
Now draw the enveloping cone to the sphere centre B of 1½ in. radius; also the cone to envelop the sphere centre *a*, radius 1½ in.

Their intersection will readily be obtained by taking a spherical section of the two cones. Imagine a sphere described with C as centre, the intersection of the two axes; and, any convenient radius, draw the arc 4 3 2: then 4 3 will be the plan of section made through the cone axis CB, and 1 2 is the plan of section made through the cone axis C*a*. D, the crossing-point of these sections, is a point in the required line; its height above the plane of axis is DD.† Now

---

* Parabola is a curve traced by a point moving such that its distance from the directrix and the focus is always equal.

† Rotate the circle of section to the horizontal plane; then DD_1, drawn perpendicular to the diameter 4 3 to meet the circumference, will be the height of D above the plane containing the axis C*a*, CB.

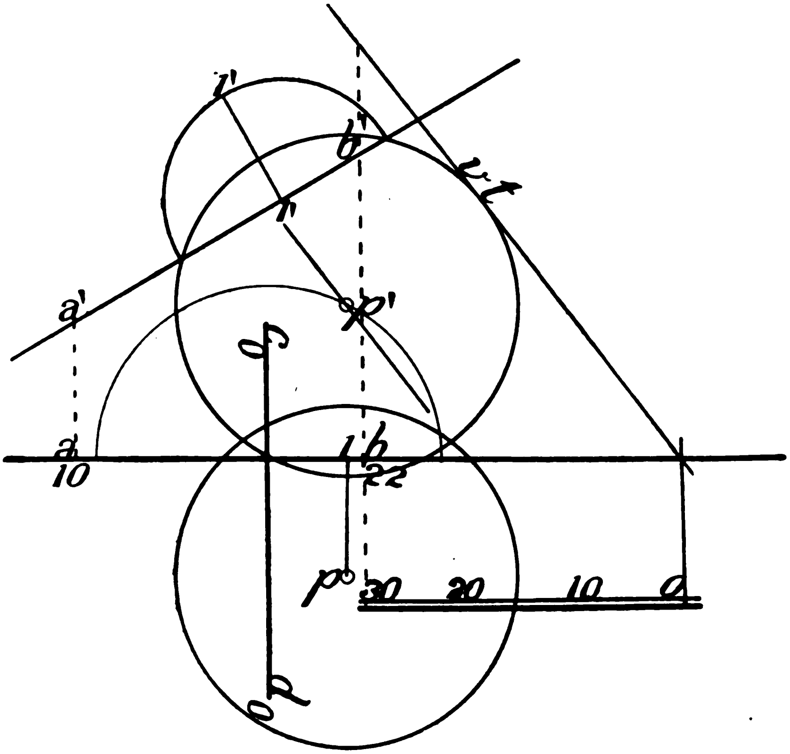

V
c'
d'
d'
c'''
X
C
c30
O
f' Y
2
6
D
3
d
D1
4
a0
6'
w
u
7
v
1
s
1'
b
B
b14
x
u
x
f
F
H

rotate the whole back to their original positions. The point *d′* will be perpendicularly above the plane, at a distance equal to DD^1; its plan is at *d*. Draw the line *c′d′*, *cd* having its horizontal trace at *f*, which is on the intersection of the bases of the cones as before; hence *cdf*, *c′d′f′* will be the required line. This second method is recommended, it being much simpler, having no elliptic or parabolic curves to draw.

This question is possible only when the points are sufficiently near so that the cones may intersect.

No 54, 1891. **Two lines, AB and CD, are given by their figured plans (the latter is in the horizontal plane), and a plane is given by its scale of slope. Determine a sphere of 1¼ in. radius touching the lines and plane (unit 0·1 in.). (The sphere is intended to be below the plane.)**

The lines marked *a*10, *b*22, *co*, *do*, and the plane are given. Commence by determining the elevation of the line *ab* as *a′b′*; make also the elevation of the plane as *vt* (note the line *ca* is parallel to the H.T., and line *ab* perpendicular to the H.T. of the given plane respectively). Now determine the elevation of a line parallel to the plane, and 1¼ in. away from it and from line *cd*, as shown at *p′*. With *p′* as centre, radius 1¼ in., describe the circle, cutting the line *a′b′* in two points. Horizontal distance of *p* from the vertical plane of *ab* is equal to the radius 1′1′; measure, therefore, the distance 1*p* in plan equal to 1′1′. *p* is the plan of the centre of sphere, for the line *a′b′* is tangential to the sphere at the point 1 on the rabatted circle cut by the imaginary plane containing *a′b′* and parallel to *cd*.


l'
b'
vt
r'
a'
p'
c
a
10
1
b
22
p
30
20
10
0
d

No. 56, 1892. ***ab* is the plan of an edge of an octahedron; the plan of the adjacent edge falls on *a*C. Complete the plan of the octahedron.**

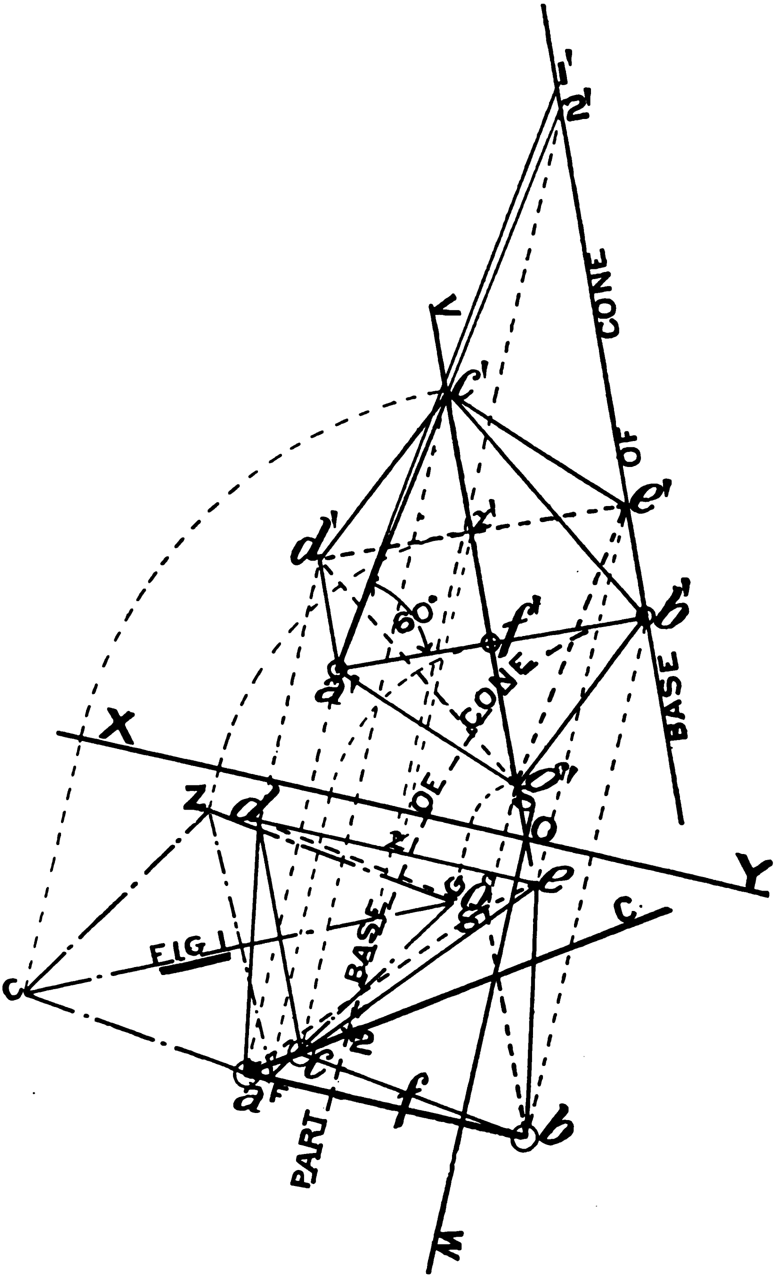
Let the lines *ab*, *ac* be given, *a*10, *b*20, which are to be two edges of the octahedron.

We may commence by taking XY parallel to *ab*, and determine the elevation *a'b'* of *ab*. Now consider the line *a'b'* as the axis of a cone semi-vertical angle of 60°,* the plane of base passing through *b'*, meeting the generator *a'*1' in point 1'. The plan of base of cone intersects the given line *a*C in point $2_1 2'$. Then the lines *ab*, *a'b'*, *a*2, *a'*2', contain an angle of 60°. Next cut off a length *ac*, *a'c'* equal to *a'b'*; thus, make an elevation of *a'*2' on *xy* parallel to *a*2, when it is seen in true length: then the △ *a'b'c'* is equilateral. Now, the plane VOW perpendicular to *ab*, *a'b'*, must pass through the centre of the solid, and must, of course, contain the diagonal, this plane rotated about its horizontal trace into the horizontal plane, carrying point *f'* and *c'* with it to F and C. Then FC is the true length of *f'c'*, and the quadrilateral FCZG is the true shape of section through centre of solid made by plane VOW; thus CZ, ZG, GF = FC, and FZ = *ab*, the edge of solid. Now rotate the plane back to its original position; then Z will be at *z'z*, and G at *gg'*, and from the symmetry of the octahedron, *d'e'* drawn through *z'* parallel and equal to *a'b'* will be another edge of the solid, and *g'* will be the lower vertex of it. Join all these points, *i.e.*—*a'd'*, *b'e'*, *e'c'*, *b'c'*, *d'c'*, *d'g'*, *a'g'*, *b'g'*, *e'g'*, and the elevation is complete.

The plan may readily be determined by drawing the plan of line *d'e'* through *z*, meeting the projectors from *d'e'* in *d* and *e*; now join all the points in order as in the elevation, and the octahedron is complete.

---

* The octahedron is a solid having eight faces, all of which are equilateral triangles; therefore the angle between any two of its edges is 60°.


2'
c'
CONE
OF
e'
d'
x'
60°
f'
b'
BASE
a'
T
CONE
X
Z
d
OF
o'
O
Y
e
x
L
G
g
BASE
FIG I
C
C
c
n
2
a
F
f
PART
b
W

No. 56, 1894. **ab is the plan of an edge (2 in. long) of an octahedron. The plan of an adjacent edge falls on ac. Complete the plan of the octahedron. Unit (figured heights) = 0·1 in.**

Let ab and ac be given; index of a 8, length of ab = 1·7 in. length of ac indefinite.

This question is so similar to the last, that no special drawing of it is necessary, and we may immediately commence by finding the height of b, which is readily done by erecting a perpendicular of indefinite length at b, and with a as centre, radius 2 in., describe an arc cutting the perpendicular in a point whose height added to 8 (the index of a) will be the height of the point b.

Now with this true length $a'b'$ as base construct an equilateral triangle $a'b'c'$, and imagine this to revolve about its base, when vertex c will have described a circle. And the intersection of this circle with ac will be the required length of the edge ac, which is readily seen in plan, as it must be the crossing-point of the elliptic plan of the circle with the line ac.

This method will more readily determine the length of ac than that shown in the last question. Further, the solution will be precisely similar, and need no further description.

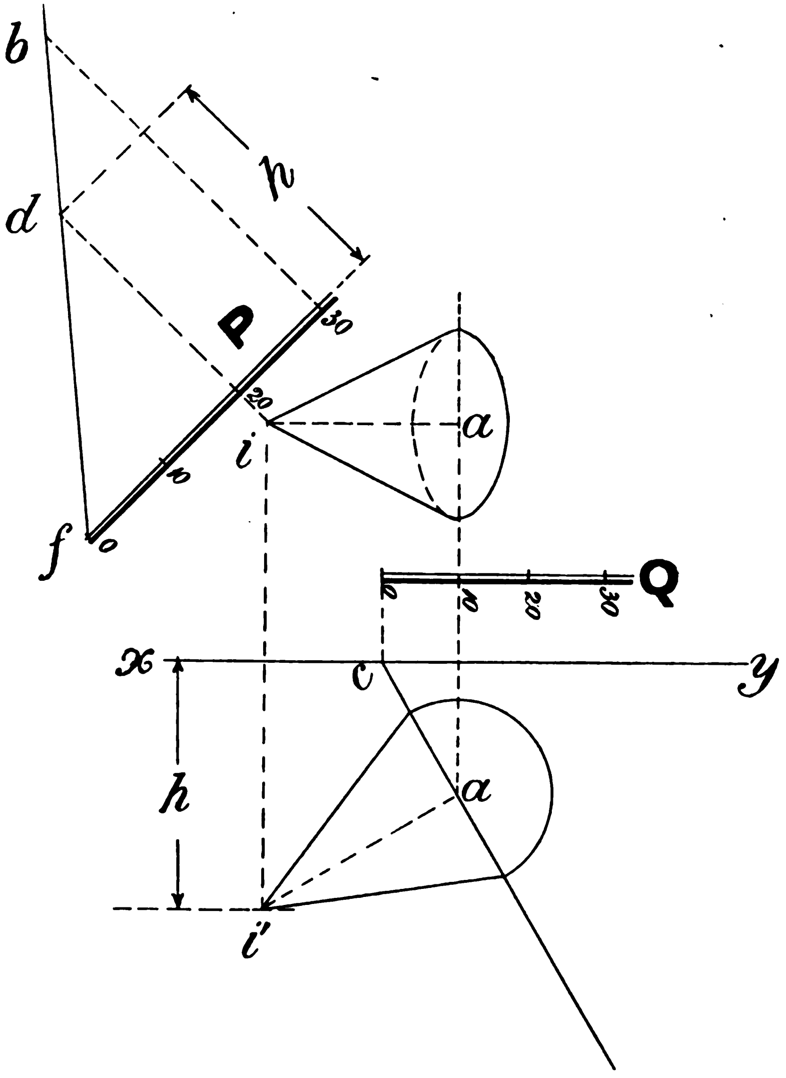
The method here shown is simpler, as the elliptic base of the cone is dispensed with. Thus, with p'' as centre and any radius, draw a circular arc cutting the generators $p''q'$, $p''q$ in points 11, and $p''r'$, $p''r$, in points 2, 2; join 1, 1, 2, 2: their crossing-points t' will be points in the required generator $p''t'a$.

The plan of point t' will evidently lie on the circumference of the horizontal circular section 1 1.

With p as centre, describe the circle which is the plan of the horizontal section 1 1, and project t' to tt.

Join pt, pt_1 and produce it to meet the circular base of the cone in points a and b. Now project a to a' and b to b' on XY, and join $p'a,'$ $p'b'$.

Then $p'a'p'b'$ is the elevation, and $papb$ is the plan, of two lines satisfying the given conditions.



57, 3 **Draw a cone, base 15 units diameter, height 17·5 units, its base in plane Q, its vertex in plane P, and the centre of its base at 10 units above the horizontal plane of projection (unit = 0·1 in.).**

Take *xy*, a new ground-line parallel to the scale of slope Q, and erect the end view *ca* of the plane Q, in which *a* is 10 units above the ground. Erect *ai'* perpendicular to *ac*, and equal to 17·5 units. The plan of *i'* will lie somewhere on a horizontal, *ii'*, drawn perpendicular to *xy*. Get *fb*, the end elevation of the plane P, and set up *h* perpendicular to scale of slope P, and obtain the point *d* in the plane at the height *h* above the ground. The vertex of the cone will lie in a horizontal through *d*. The intersection *i* of the loci of the vertex, is the plan of the actual vertex, and *ai* is the plan of the axis of the cone.

No. 60, 1893. ***abc* is the plan of the face of a tetrahedron ABCD. Its fourth vertex D is at a height 9. The plane angle BAD is 35°, the plane angle CAD is 30°. Find the plan *d* of the vertex D, and complete the tetrahedron (unit = 0·1 in.).**

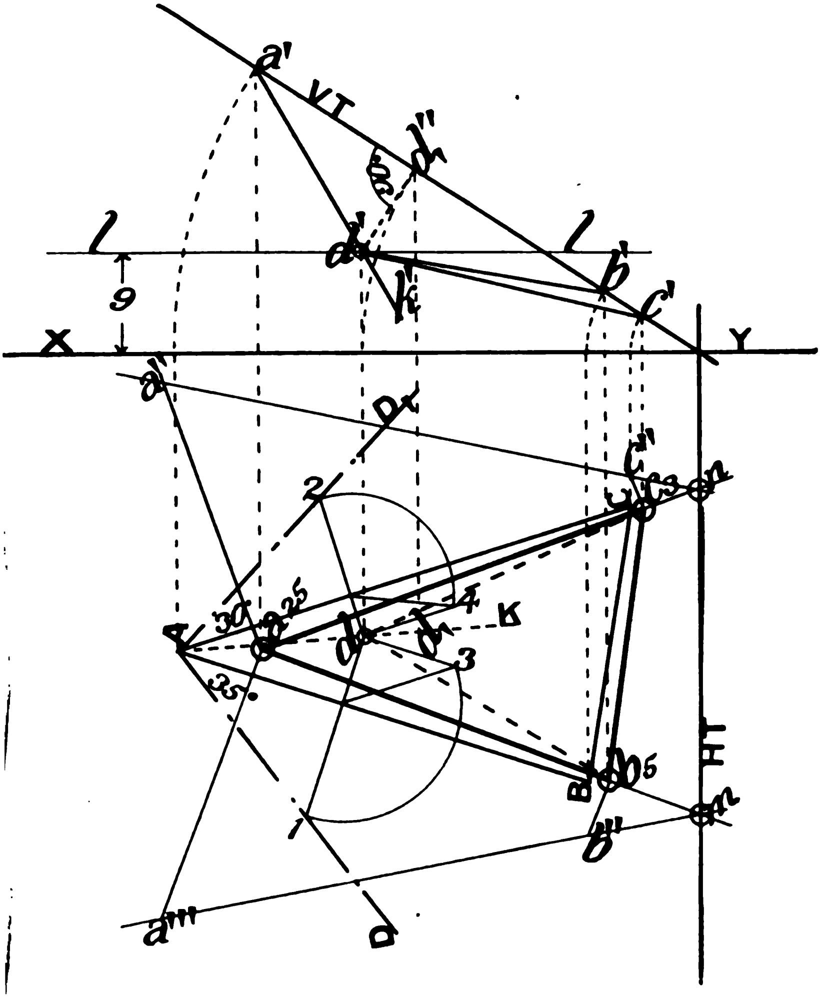
Let the △ *abc* be given; a_{25}, b_5 c_3, *ab* = 1·7 in., *ac* = 1·8 in., *bc* = 1·2 in.

Determine the plane containing the given △ *abc*, shown by its traces VT, HT. Now rabat the plane to the H.P., carrying the triangle *abc* with it to ABC; then ABC is the true shape of one face of the solid.

Next draw the line AD_1 at an angle of 30° with AC, and AD at an angle of 35° with AB; consider these as the two sides meeting in A, developed into the H.P. Then conceive these two faces as being rotated about the lines AB, AC, until their free edges AD_1, AD meet at the line AK. To do this, select any points in AD, AD_1, as 1 and 2, at equal distances from A; then during rotation these points will travel in vertical planes whose H.T.'s are perpendicular to AC and AB respectively, intersecting in point *d*. Then the line A*d*K is the edge of the solid common to the two faces BAD, CAD. Now we have only to cut off the line AK in point D so that its height is 9. To do this, we must determine the perpendicular distance of *d* from the plane of *abc*, as shown at d_4 or d_3, and, rotating the plane back to its original position, *d* will come to *d''*. Draw the perpendicular *d''d'* equal to d_4 or d_3, and draw the line *a'k'*; then *a'k'* is the elevation of the indefinite edge AD of the tetrahedron. The point D will be determined readily, as it is the intersection of the edge *a'k'* with a horizontal plane *ll*, level 9,*

---

* *If line a'k' were taken above the plane, d would be on the right of the plane, giving another solution.*



which in the present case happens to fall on point *d'*: then the projection of the solid may be completed by joining the vertex *d'd* to the angles of the base *abc*, *a'b'c'*, which fulfils the required condition. The student is strongly recommended to thoroughly realize the method adopted here for finding the line AK by conceiving the two faces BAD, CAD as developed to the horizontal plane, and then rotated back till they meet, as it will be found useful for the solution of spherical triangles.[*]

---

[*] See Angel's "Geometry," part ii. chap. xvii. p. 314.

No. 57, 1891. **Determine the right circular cone, the semi-vertical angle of which is 25°, and of which the given lines VA, VB are generators.**

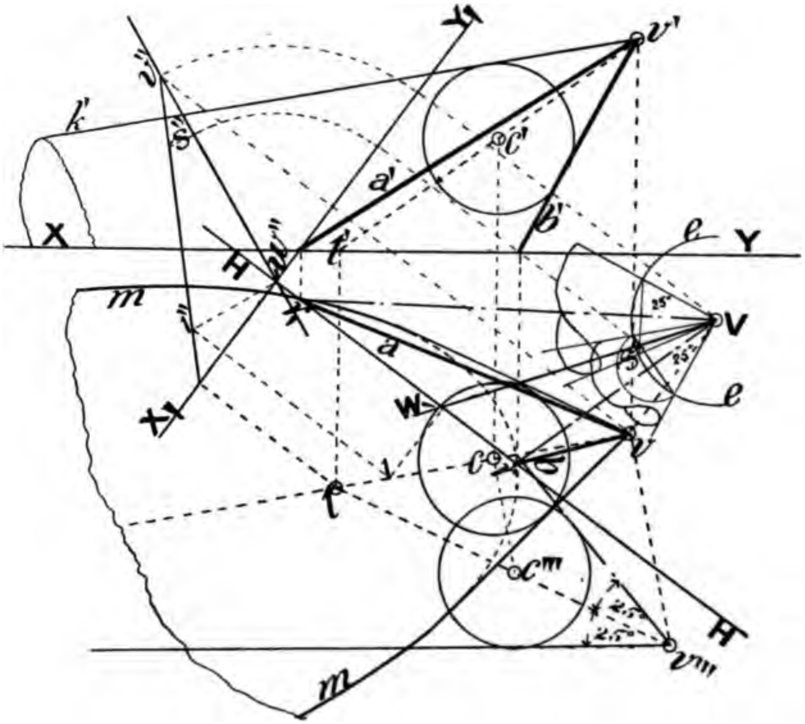
The given lines are shown in thick lines.

Determine the plane containing the two given lines *va*, *vb*, its horizontal trace HH passing through r and r_1, these points being the traces of *va*, *vb* respectively. Now rotate this plane about its H.T. until it comes to the horizontal plane, carrying the lines *va*, *vb* with it, which are shown by the chain lines Vr, Vr_1.

Let us consider these lines, so rabatted, as the axes of two cones the semi-vertical angle of which is 25°; the generator of intersection of these two cones, such as VW, will be the axis of the required cone.

This intersection may be found readily by taking a spherical section *ee*, centre V, whose plans are straight lines intersecting in S. The height of S above the plane of V rVr_1, the axis, is obtained by taking a perpendicular to the diameter of the section rabatted to the horizontal plane, from S to the circumference. Now rotate the plane and lines Vr, Vr_1 back to their original position as shown at $v_1'w''$: the axis VSW is contained by a plane which is perpendicular to the plane containing the lines. Determine the line S$''$ perpendicular to this plane, and of the same height as S in the plane of section so rabatted. Draw v''S$''$; then v''S$''$ is the elevation of the required axis on X_1Y_1 perpendicular to HH, the trace of the plane. Produce v''S$''$ to 1$''$, and make w''1$''$ perpendicular to $w''v''$. The plan of w''1$''$ is W1, 1$''$ being a point in the axis; v'' is another point. The line drawn through v1 is the plan of the required axis, having its horizontal trace at point t.

Consider the axis $v1t$ in plan as ground-line, and determine its elevation, tv'''; next draw the generators of the cone, semi-vertical angle of 25°, and determine the elliptic trace of its base on the horizontal plane.



Its plan and elevation relative to the given XY may readily be obtained by inscribing a sphere centre c''' in the cone, and determining its plan c and its elevation c', and drawing the sphere. Now, the generators enveloping this sphere are the generators of the required cone, having *va*, *vb* also as generators as seen in plan and elevation, fulfilling the given conditions.

It will be seen that only a part of the elliptic base of the cone has been drawn in the figure for want of space, but the student can easily follow the working and determine the complete plan and elevation for himself.

No. 59, 1894. **A right cylinder (diameter $1\frac{1}{8}$ in.) touches the two given spheres whose centres are *a* and *b*. The axis of the cylinder passes through the given point *p*. Determine the position of the cylinder. Unit (figured heights) = 0·1 in.**

The spheres centres *a* and *b* are given, together with the point *p*. Index of *a* 23, *b* 10, and *p* 14.

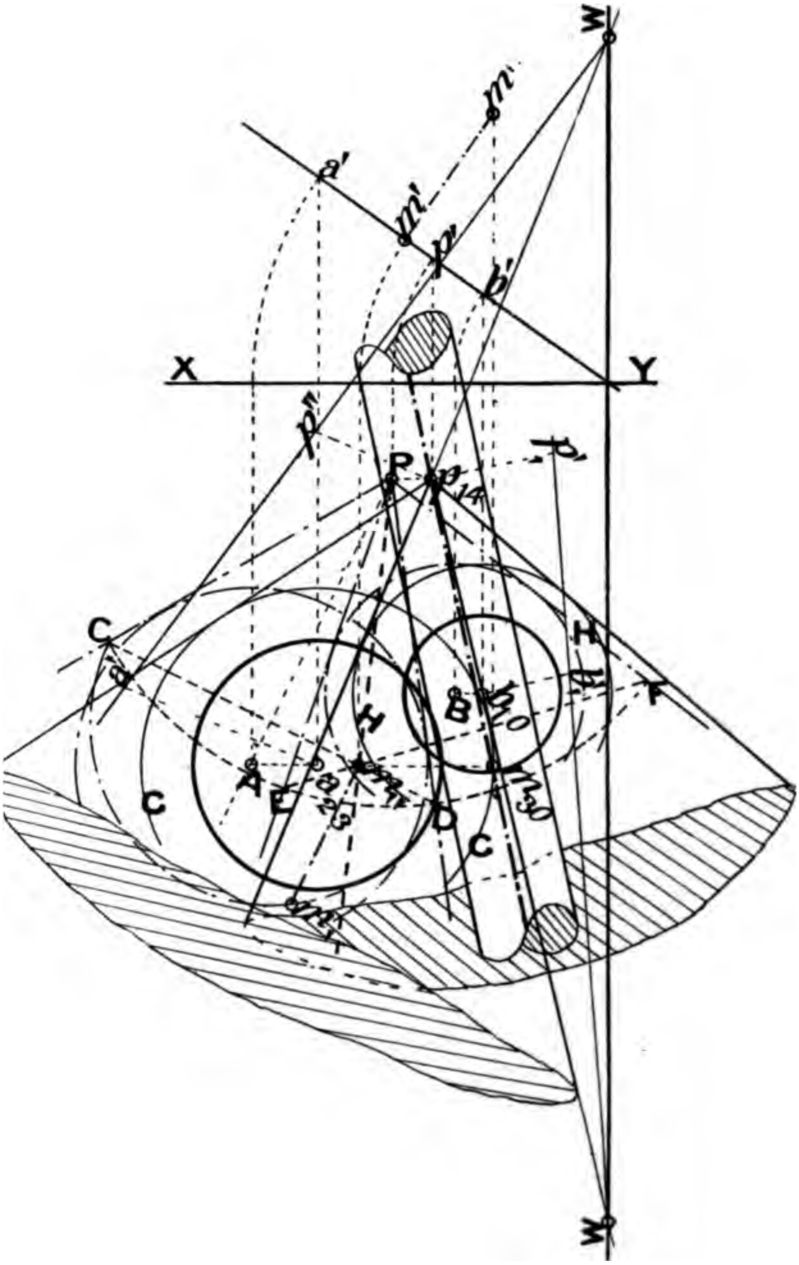
Before proceeding with this problem, it would be well to analyze the given conditions. We see that the axis of the required cylinder has to pass through the given point *p*. And evidently it must pass the given spheres at a distance of $1\frac{1}{8}$ in. ÷ 2 away from their surface. Suppose, then, that each sphere is shrouded by another sphere of radius increased by $1\frac{1}{8}$ in. ÷ 2. And assume the point *p* as common vertex of two imaginary cones shrouding these enlarged spheres. Then the intersection of these two cones will satisfy the required conditions, and will therefore be the axis of the required cylinder.

To do this, join *pa*, *pb*, and *ab*, and describe the spheres GG and HH, centres *a* and *b*, with radius increased by $1\frac{1}{8}$ in. ÷ 2, and draw the outside generators of the cones.

Now determine the plane whose horizontal trace is WW, which contains the lines *pa*, *pb*, the axes of the cones respectively; and draw the vertical trace relative to XY which is perpendicular to WW at any convenient position, and determine the elevation *a'b'p'* of *abp*, as shown.

Next rotate this plane about its H.T. till it reaches the horizontal plane, carrying the points with it to ABP; and draw the cones in this position, as shown at PC, PD, PE, PF.

Now their intersection may be readily found by taking a spherical section of both cones; thus, with P as centre and any radius PC, describe the arc CEF, cutting the generators in CD and EF; join CD and EF. Their crossing-point m_{11} is a point in the plan of the required axis.


W
m'
a'
m'
p'
b'
X
Y
p''
P
p
14
p'
C
d
H
H
F
B
b
0
H
A
E
a
D
m
0
C
C
am
W'

The height of point m_{11} is found by rotating the circle CD (or EF) about its horizontal diameter into the H.P.; and draw $m_{11}m$ perpendicular to CD, cutting the circumference in m_1. Then $m_{11}m_1$ is the perpendicular height of m above the plane containing the axes of the cones.

Now rotate the plane back to its original position, carrying m_{11} to m', and erect $m'm''$ perpendicular to the plane and equal to $m_{11}m_1$. The plan of m'' is at m; its height above the H.P. is seen in the elevation above XY, and is 30 units. Join pm; then pm is the axis of the required cylinder index of p 14 and m 30.

Complete the cylinder by drawing the generators parallel to its axis pm and $1\frac{1}{2}$ in. ÷ 2 on each side, as shown.

Its position only is required, but the student may make an elevation on a plane parallel to its axis, and complete its elliptic trace on the H.P. This problem is very similar to No. 48, 1887.

No. 57, 1895. **O is the centre of a sphere resting on the horizontal plane, shown in plan by a circle in the diagram, *p* is a point on the sphere above its centre, *cd* is a given line.**

1. Find on *cd* a point *v* so that the cone described with *v* as vertex, circumscribing the sphere, shall contain *p* on its curve of contact with the sphere; and draw that curve.

2. Draw a cube in the cone, one face tangent to the sphere, perpendicular to the axis of the cone, and with one horizontal diameter; the opposite face having its four angles contained on the surface of the cone. (Unit = 0·1 in.)

1. The circle in plan, centre o, and the lines cd, $c20$, $d40$, are given.

The required point, v, will be the point of intersection of a tangent plane to the sphere at point p with the given line *cd*; for every generator of a cone is perpendicular to the radius of the sphere which it envelops at its point of contact; and a tangent plane to a sphere is, of course, perpendicular to the radius at its contact point.

Join *op*, and take this as a new ground-line; erect *pp″* perpendicular to *op*, meeting the circumference of the sphere at *p″*; join *op″*. Then *v″p″u*, perpendicular to the radius *op″*, will be the vertical trace of the tangent plane to the sphere at point *p* relative to the ground-line *opu*. Then the line through *u*, perpendicular to *opu*, will be a horizontal line on the plane, which is level with centre of sphere, height 10.

Now draw level 20, say, or any other level which may be convenient on the plane, to meet corresponding level on the line *cd*, and drawn perpendicular to it, meeting as at *w*; the meeting-point of levels 10 fall outside the paper. The line drawn through these points will be the intersection of these two, *i.e.* the plane of line *cd* and the tangent plane. The crossing-point *v* of this intersection and line *cd* will be the vertex of the required cone,* and the generator *vp* is contained by the plane, and is perpendicular to the radius *op*. Join *vo*, which will be the axis of the cone.

Next, on a ground-line *xy*, parallel to the axis *vo*, make an elevation of the cone. [The height of *v* is found at *v″* on the tangent plane above the centre of the sphere.] Draw the circle of contact, *g′g″*, which is a straight line in this elevation, and the ellipse *gg* in plan, which will pass through *p* as required.

2. Cut the cone by a plane tangential to the sphere and perpendicular to the axis of cone, as at 66′. Now inscribe in this cone a square pyramid, thus: Anywhere, as in fig. M, draw the plan of the base 66′, and inscribe a square in it. Make an elevation efv_1 of the pyramid on a ground-line, *ef*, parallel to one of its sides, and in this

---

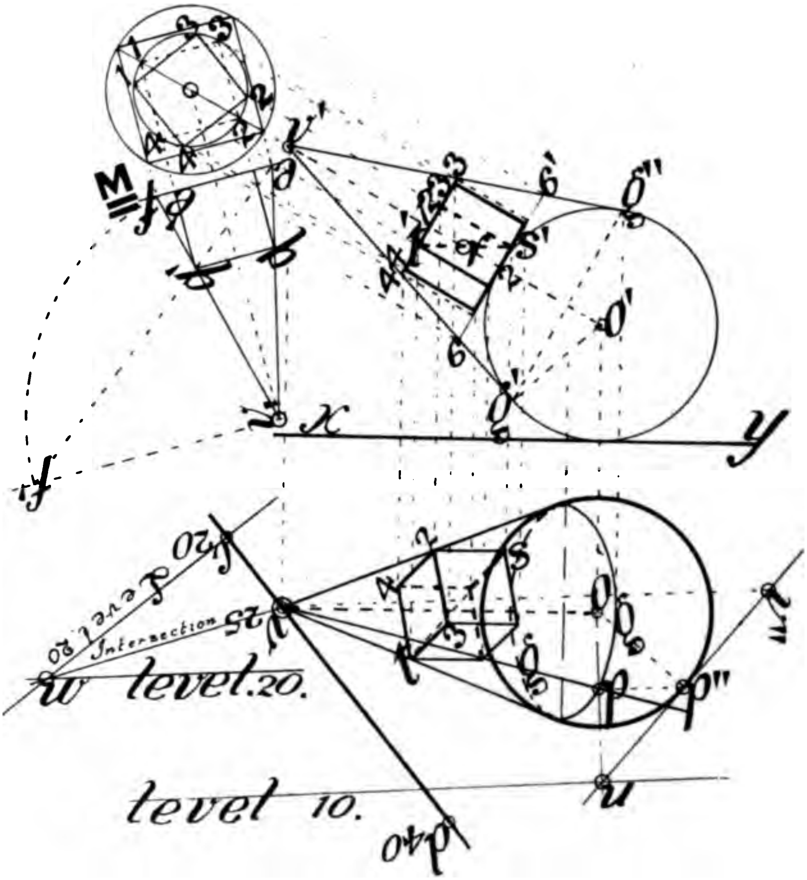
* A rather quicker and simpler way of getting *v* is to draw an elevation of *cd* on *opu* as ground-line. The intersection of this elevation and *v″p″u* is the elevation of *v*.

triangle inscribe a square, which is easily done by drawing v_1f' parallel to ef, and with v_1 as centre, radius v_1f, describe an arc cutting v_1f' in f'. Join $f'e$, cutting v_1f in q', and draw $q'q$ parallel to ef. Then $q'q$ is a side of the inscribed square. Now, as this square is inscribed in one vertical section of the pyramid, evidently there will be a similar square in a plane perpendicular to it, in a companion vertical section; these, being moved parallel to themselves till they meet the edges of the pyramid, will form a cube, having its four upper angles in the four edges of the pyramid, which, of course, are generators of the cone, shown in the plan fig. M by dotted lines 1, 2, 3, 4. Next project these back to the elevation on xy as shown, and build the cube, which will have one face in the plane tangential to the sphere, and perpendicular to the axis. Now determine the centre r' of the cube, and draw the line $s't'$ parallel to xy, meeting the planes of the ends of the cube in $s't$. We must now revolve the cube about its axis (*i.e.* the axis of the cone of which the cube is inscribed) until the two opposite corners of the cube come to s', t', as one diameter * is to be horizontal.

Project s' and t' to 2 and 1 on the circle circumscribing the base of the cube, fig. M, and complete the square shown in full line. Now project back to the elevation on xy, and complete the elevation of the cube. The plan is obtained in the usual way, *e.g.*, the point t' is in front of the line 3, 4 (fig. M); similarly S′ is as much behind the axis as it is behind the line 3 4, and so on for each point. The points so found being joined in order, the plan and elevation of the inscribed cube are complete, with one diameter, $s't$, horizontal, as required.

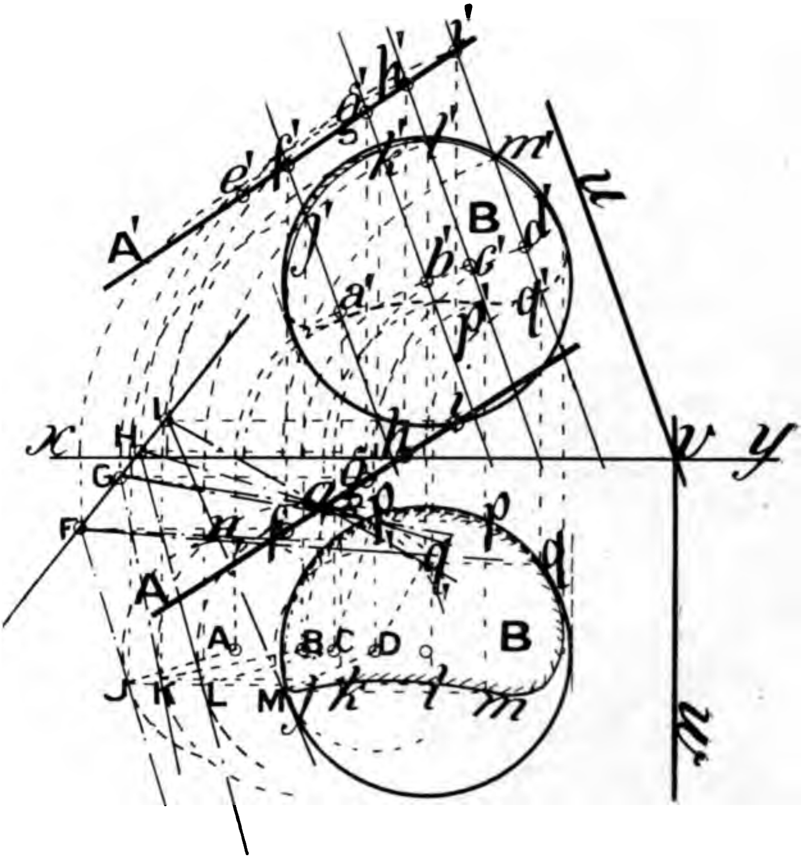
---

* "Diameter" is here used indiscriminately. The writer prefers using diagonals for angular objects, and diameter for spherical and circular objects.


1
2
3
4
v'
M
f
a
b
v
x
y
f₁
6'
6''
s'
o'
6
level 20
Intersection
f20
v25
w20
w
level.20.
level 10.
u
d40
s
o
p
p''
t
r

No. 56, 1895. **AA′ is a given line, BB is a sphere, and *uvw* the traces of a plane. A surface is generated by a straight line moving along AA′, touching the sphere, and remaining constantly parallel to the given plane. Find the trace of the surface on the sphere in plan and elevation. Indicate distinctly the visible and invisible portions.**

The line which generates the surface will be tangential to the circular sections made by planes parallel to the given plane. Therefore, take a series of planes parallel to the given plane, cutting the sphere in circles, and the line in a point. The first plane shown cuts the line in point *i′*. Rotate this plane to the horizontal plane, carrying the point *i′* with it, and also the circular section of which *d′* is the centre to I and D. Then from I draw IM, Iq_1 tangential to this circle of which M and q_1 are the contact points. Rotate the plane back to its original position, carrying the points M and *q* with it to *m′* and *q′*. Then *m′q′* and *mq* will be points on the curve traced by the generating line on the sphere. Similarly proceed with the other planes, passing through *hgf*, etc., till a sufficient number of points are found, and the curve drawn freehand through these will be the trace on the sphere. The invisible portion in plan and elevation is shown dotted.


i'
g' h'
e' f' s' k' l' m'
u
A'
j'
B
d'
b' c'
a' p' q'
I
h
x H
v y
G
o
F
n
p
p
q
q
A
A
B C D
B
J K L M j k l m
w

No. 60, 1894. **The elevation of a right cylinder, whose axis is parallel to the vertical plane of projection, is given. The cylinder penetrates the given sphere, and also touches its surface internally. Draw in plan and elevation the complete curve of intersection.**

The sphere and cylinder shown in elevation above XY are given.

The axis of the cylinder is given (in the question) only in direction, and before the curve of intersection can be found its position must be fixed, so that it may touch the sphere internally.

This may be done by taking a ground-line X_1Y_1 perpendicular to the axis of the cylinder, and determining the plan of both sphere and cylinder (shown at fig. 1).

The plan of sphere will be the plan of the great circle passing through a', and the plan of the cylinder will also be a circle, and has to be moved between its projectors from its elevation till it touches the great circle of sphere in a point at a.

Then aa' is the point of contact of cylinder with sphere internally, and therefore we have fixed the cylinder in position. "It will, however, be seen that the cylinder might be so placed as to touch the back surface of the sphere."

Now take a series of sections of the sphere and cylinder perpendicular to its axis, shown at ee' . . . ff', etc. (fig. 1). Each of these circles cuts through the cylinder in two points, such as b and c, which are points in the curve of intersection. Project each of these points so found to their respective circular sections in elevation, thus, a to a', b to b' and b'', c to c' and c'', and so on, till a sufficient number of points are found, and the curve drawn freehand through these points will complete the elevation, which is shown in two lips, one above a' as $a'b''c''$, etc., and the other below a' as $a'b'c'$, etc.

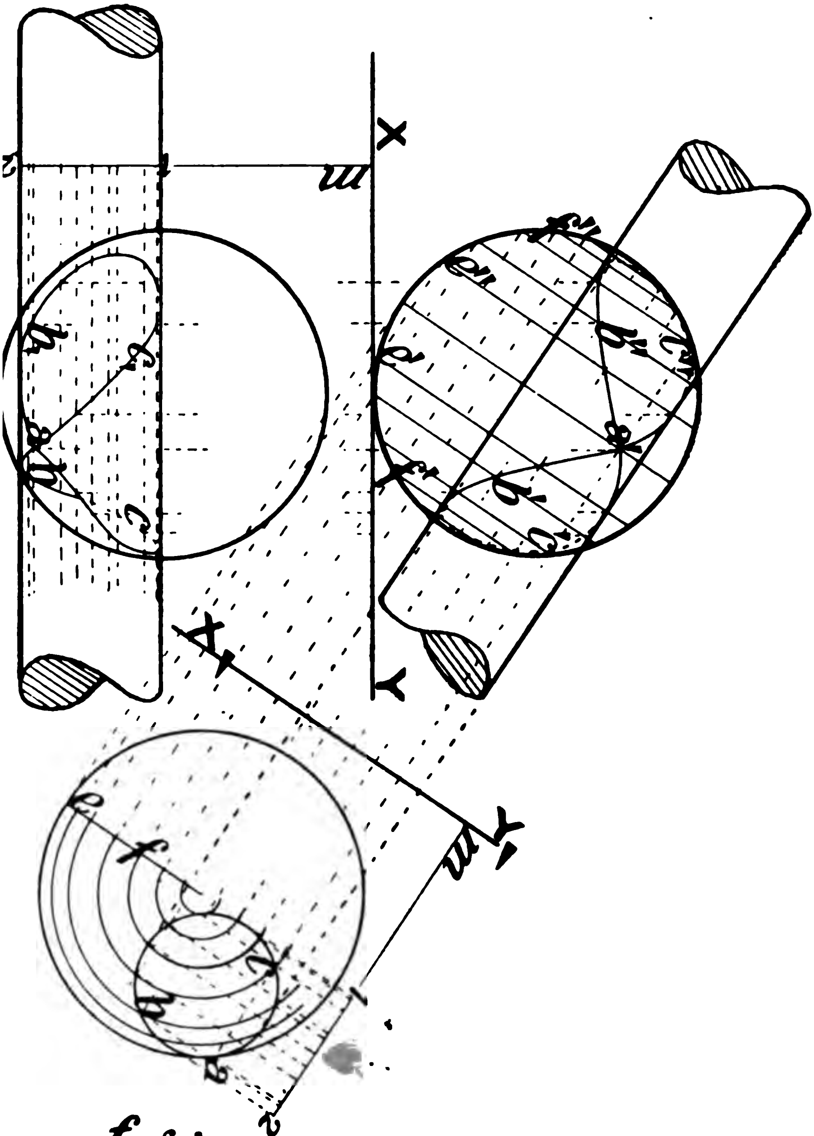


fig. 1.

Next determine the plan of the whole, when the cylinder is to be parallel to the vertical plane of projection.

Take the distance *m* to 1 and each of the distances between 1 and 2 (Fig. 1), and transfer them to *m*, 1, 2, vertically under the given XY, *i.e.* the distance of every point on sphere or cylinder in front of X_1Y_1 will also be the same in front of the given XY. Complete the plan of sphere and cylinder as shown.

Now project each point in elevation to its proper position in plan; thus, a' to a, b'' to b_1, c'' to c_1 b' to b, c' to c, etc., till a sufficient number of points be obtained and the curve drawn through each, when the curve of intersection is completed both in plan and elevation.

No. 56, 1896. **Ev, $E'v'$ are the projections of the axis, and v, v' those of the vertex of a cone, whose generating lines are inclined at 20° to the axis; CD, C′D′, the projections of the axis of a cylinder of $\frac{3}{4}$ in. radius. Draw the projections of a sphere of $\frac{5}{8}$ in. radius, touching both solids, its centre in the vertical plane represented by KJ. Show the projections of the points of tangence of the solids.**

The lines given on the examination paper are reproduced on the top of the accompanying figure, and transferred to the main figure below.

If the cone is enveloped by another cone whose curved surface shall be $\frac{5}{8}$ in. from that of the given cone, this enveloping cone surface will contain the centres of all spheres touching the given cone.

Similarly, the enveloping cylinder will contain the centres of all spheres touching the cylinder. The intersection of the enveloping cone with the enveloping cylinder will give the locus of sphere-centres in which the sphere can touch both solids.

First describe the trace of the enveloping cylinder in the ground. This trace will be the ellipse whose centre is C.

Take a new ground-line, X_1Y_1, parallel to the plan of the axis of the cone, and draw the elevation of the enveloping cone, the elevation of the given cone being Nv_1M. The trace of the enveloping cone in the ground is the remaining ellipse.

Now cut the cone and cylinder by a series of planes parallel to the axis of the cylinder and containing the vertex F of the cone. Through F, F′ draw the projections of a line parallel to the axis of the cylinder. Its horizontal trace is at H. As the cutting planes are parallel to the axis of the cylinder and contain the vertex of the cone, they will all contain the line FH, and consequently their horizontal traces will all pass through H. Therefore draw any line H*ie*, this line being the horizontal trace of a cutting plane. The generator in which it cuts the surface of the cone is *e*F, and the generator in which it cuts the surface of the cylinder is *ic* parallel to the axis of the cylinder. The generators *ic* and *e*F intersect in *c*, therefore *c* is the plan of a point in the locus of sphere-centres. In this way the locus *gh* is obtained. The intersection *n* of JK and the locus *gh* is the plan of the centre of the sphere required.

To get its elevation, draw the generator of the cylinder passing through *n*; it cuts the trace of the cylinder near *i* (generator not shown). The new elevation of this generator is n_2n_3, the point n_2 being the actual elevation. The elevation on X_1Y_1 as ground-line is at n_1, and its elevation on XY as ground-line is n'. The point n_2 lies on the circumference of a circle of the same diameter as the enveloping cylinder. It is shown in end view at n_3, and *d* is the end view of the point of contact of cylinder and sphere, u_2 being the elevation of this point on X_2Y_2 as ground-line, and *u* its plan; while u' is its elevation on XY as ground-line.

The point n_1 lies on a circle whose radius is *ts*, and whose plane is perpendicular to F_1E_1. If n_1 were rotated about the axis of cone to *t*, then *tm'* would be the radius of the touching sphere, and *m'* the point of contact. Rotating back again, the normal *tm'f* comes to n_1r_1f, and *m'* to r_1; therefore r_1 is the elevation of the point of contact on X_1Y_1 as ground-line. Its plan is at *r*, at a distance r_1w from the axis of the cone, *i.e.* $xr = r_1w$. Its elevation is at *r'* on XY as ground-line.

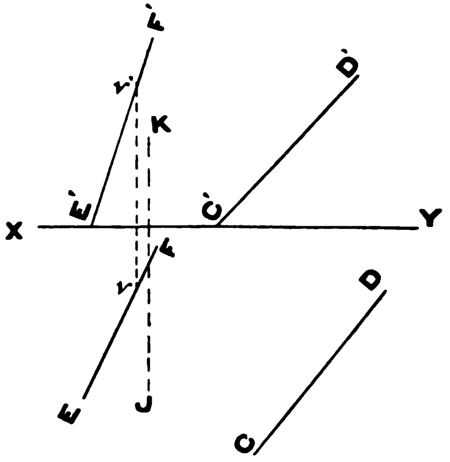
The student is recommended to work this problem to a much larger scale than that given in the examination paper. With the same initial lines, take the radius of the touching sphere as 1 in. instead of $\frac{5}{8}$ in.

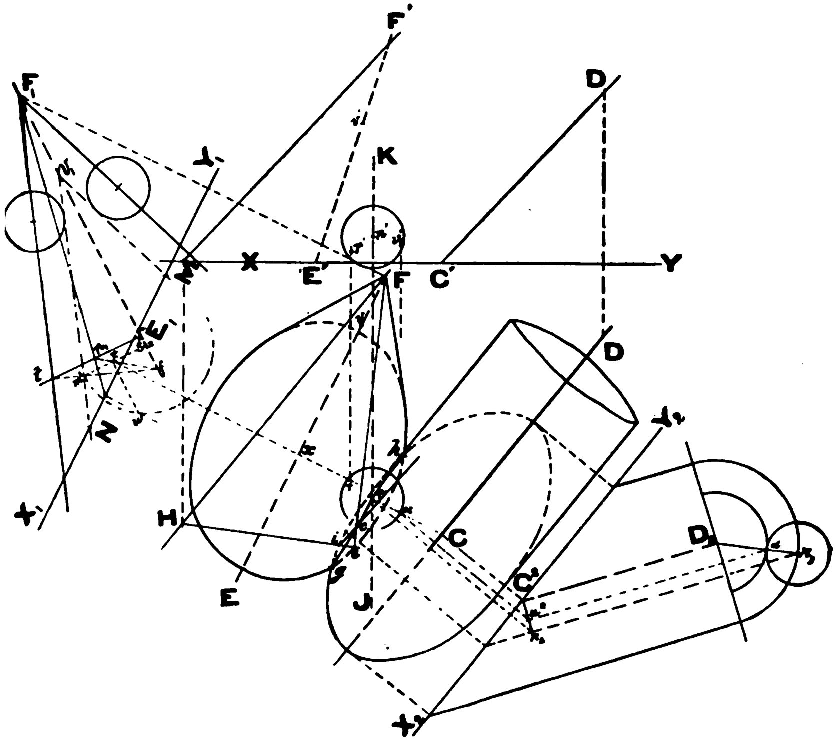
No. 55, 1896. **The plan *bd* of a line is given, together with its horizontal trace *b* and the horizontal trace *c* of a vertical line. The line BD is inclined at 50° to the ground.**

Draw the plan of a dodecahedron of 1 in. edge, one edge in the line BD, and the centre of the figure of the solid on the vertical line through *c*. Of the two possible cases, select the lower position.

Draw the plan of a dodecahedron (Fig. 1); the pentagon 1 2 3 4 5 will be horizontal, and the corresponding pentagon 16 17 18 19 20 will also be horizontal and alternate with the former one. The line 1 2 is also made to coincide with *bd*.

The lines 1 6, 17 7, etc., will all be radial, c_1 being the centre of the figure. As the side 1 2 is horizontal, there will be a diagonal of the pentagon 1 2 8 7 6, which will be parallel to it and still full size. The length of this diagonal is 5 3. Project vertically through 5 and 3, cutting 1 6 in 6 and 2 8 in 8. The diagonal 6 8 is thus located. With c_1 as centre and radius $c_1 6$, describe a circle which will contain the points 8, 9, 10, etc.


F'
v'
K
D'
E'
C'
X
Y
f
v
D
E
J
C


F'
F1
D
Y1
v'
K
X
Y
E'
F
C'
M
E
N
X1
H
D
Y2
C
D1
C2
E
J
X2

To get the height of the solid in elevation, produce 2 3 to *i*, and draw through 9 and 10 lines parallel to 2 3. With *i* as centre and radius equal to the distance between 1 2 and 4, describe an arc cutting 9*k* in *h*. Join *ih*, cutting 10*t* in *m*; then *tm* is the height of the points 9, 11, 7, 13, and 15, while *hk* is the height of the points 10, 8, 12, 6, and 14. The length 12 4 = 7 17.

The elevation of the solid on x_2y_2 as ground-line perpendicular to *bd* is shown in Fig. 3. The centre of the solid is c'', and the point *o* is the elevation of the edge 1 2. The elevation Fig. 2 is not necessary, but has been drawn to show the nature of the solid. The centre of the solid in end elevation is c_1'', whose distance away from the edge 1 2 is oc_1''. Draw an arc sc'' with this radius and *o* as centre; then, however the solid is rotated about the edge 1 2, its centre will lie in sc''. It also lies in the vertical through *c*; therefore it must be at c'' in elevation, and of course at *c* in final plan. In this way the solid must be rotated about the edge 1 2 to its final position through the angle $c_1''oc''$. For the sake of getting heights in the front elevation, set off an angle $4oj = c_1''oc''$ in the backward direction; then the distance of a point from *oj* will be the distance of the same point from $b'e$1 2 in the elevation Fig. 4. Therefore set off *ef* equal to the distance of c_1'' from *oj*, and through *f* draw fc' equal to $b'e$, cutting the vertical through *c* in c'; then c' is the centre of the solid in elevation. The dimensions parallel to $b'e$ may be taken direct off Fig. 1 parallel to 1 2. The elevation Fig. 4 can then be easily constructed. As the elevation Fig. 3 has to be rotated through the angle $c_1''oc''$ to its final position, we may obtain the same effect by setting off the angle $uos = c_1''oc''$, and then the distance of any point from *po* will be the distance of the same point from *bd* in Fig. 5. A point has the same number in all views. This problem is long and tedious, though not difficult.

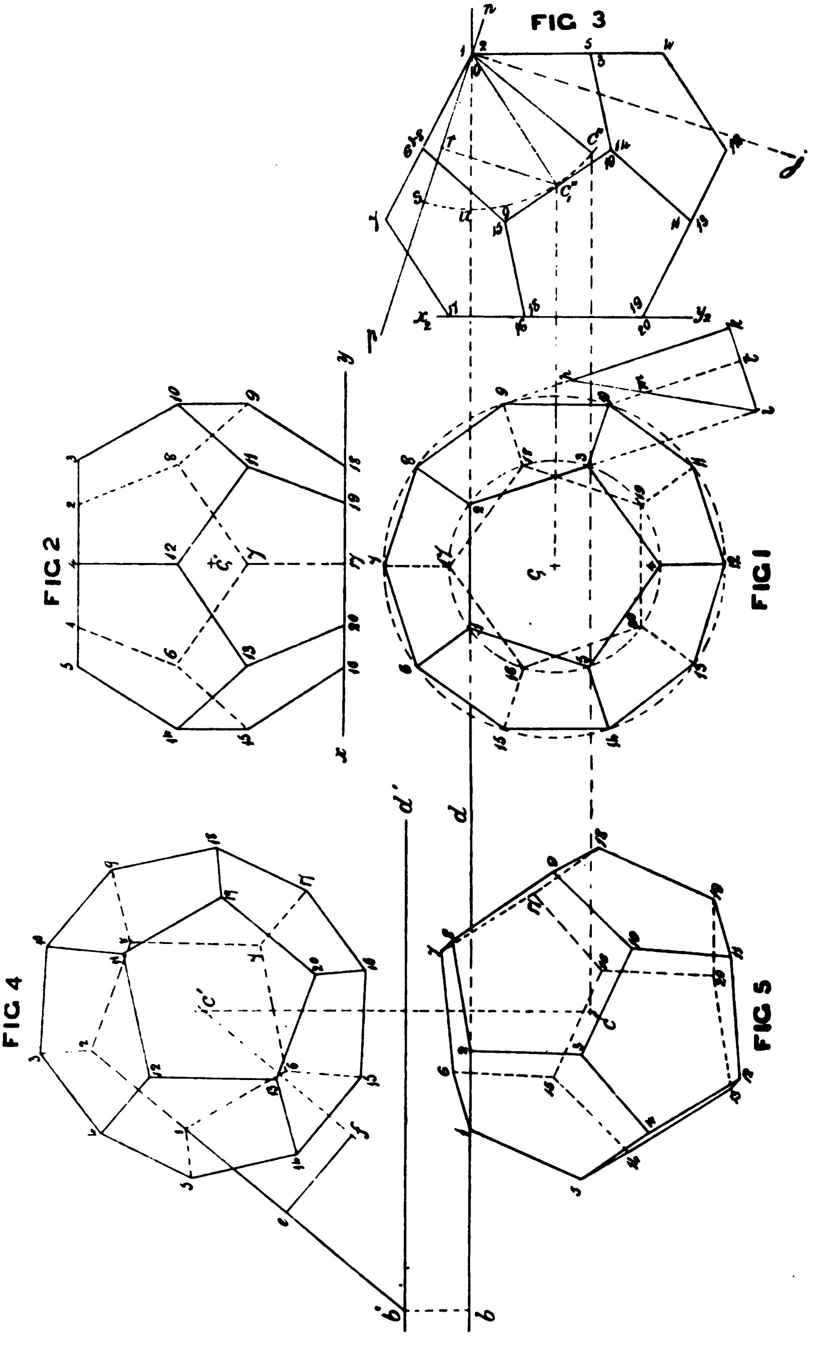


FIG 3

FIG 2

FIG 1

FIG 4

FIG 5

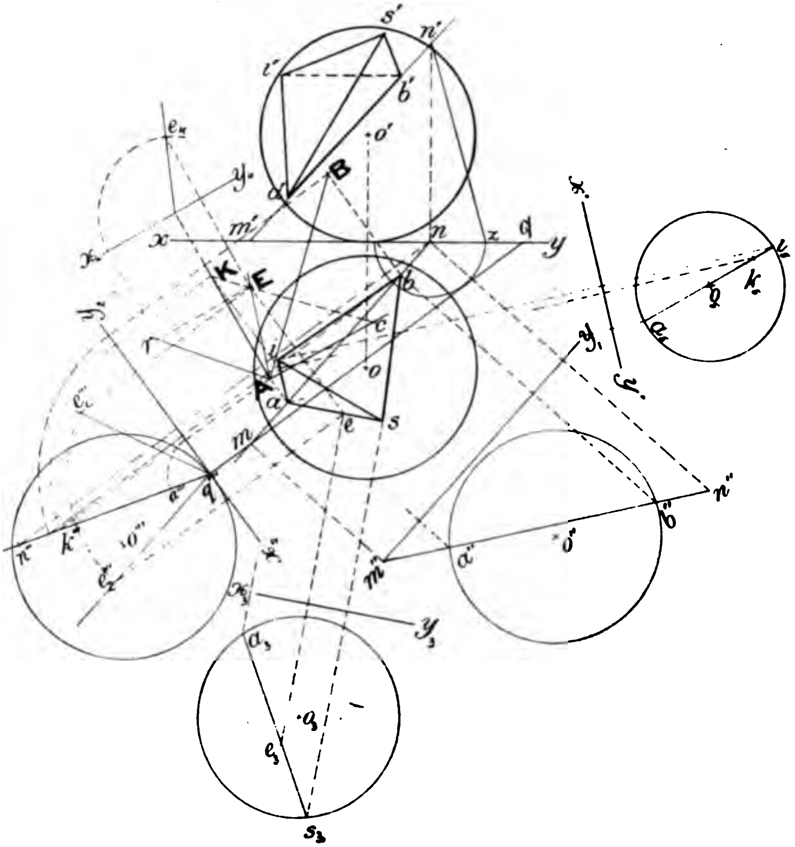
No. 57, 1896. **The projections mn, $m'n'$ of a line are given; O is the plan of the centre of a sphere of $1\frac{1}{4}$ in. radius resting on the horizontal plane. Find the intersections aa' and bb' of the line and sphere. Let aa' be the lower intersection. $aba'b'$ is one edge of a tetrahedron inscribed in the sphere, and the angles of the three faces intersecting at A are 45°, 45°, and 40° respectively, the two faces at 45° having AB for intersection.**

One of these latter faces is inclined at 75° to the horizontal plane. Of the two possible cases, take that one where the horizontal trace of the plane of the face considered would cut the ground-line to the right of the point n, and draw the complete projections of the inscribed tetrahedron.

Find the two points of intersection aa' and bb' by taking a vertical section of the sphere through the plan of the line mn. This is done on $m''y_1$ as ground-line, the two points being a'' and b''.

Now put a plane containing ab, $a'b'$, and inclined to the ground at 75°. This is done by drawing the plan and elevation of a half-cone whose axis is nn' in the vertical plane. The horizontal trace of the plane must pass through m, the horizontal trace of mn; and it must touch the semicircle whose radius is nz. This horizontal trace is mq.

An end elevation of this plane on x_2y_2 as ground-line (which is perpendicular to mq) is $q'''k'''n'''$. Rotate that plane down to the ground about its horizontal trace mq as axis, taking with it the line $aba'b'$ to AB. Set off the angles BAc and BAK, each equal to 45°; also set off the angle KAr = 40° These are the angles at the vertex A of the tetrahedron developed. To re-form the trihedral angle at A, the line Ar must be rotated about AK as axis, and will thus trace out a cone. Make Ar = Ac, and draw through r a line rE perpendicular to AK. This line is the plan of the path of the point r. Similarly, the line Ac must be rotated about AB as axis, the plan of the path of c being cE intersecting rE in E. Then the three edges meeting at the vertex A are AE, AK, and AB. This trihedral angle has now to be rotated back, so that the face

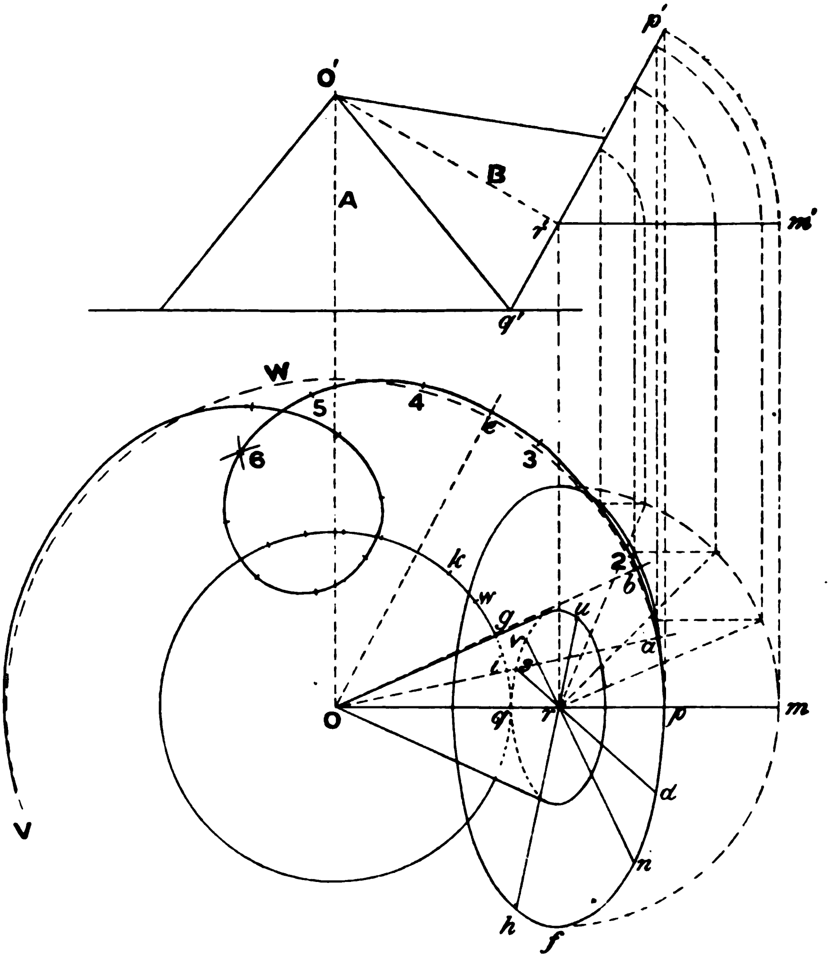


KAB will lie in the plane $q'''n'''$. This has been done at ae, ak (the point k is not labelled, because there is no room, but its position is easily seen), and ab. It is now required to locate the points i and s, where ak and ae intersect the surface of the sphere. Take a section of the sphere by a vertical plane containing ak. This section is shown on $x'y'$ as ground-line, and the point i_3 obtained.

Finally, cut the sphere by a vertical plane containing ae. The section is shown on x_3y_3 as ground-line, and s_3 obtained. Joining up the points a, s, b, i, the plan is completed. The heights in elevation may be obtained from the auxiliary elevations.

No. 58, 1892. **The elevations of two right cones, A and B, having a common vertex, are given. The cone A stands on the horizontal plane, and the cone B revolves upon it. Draw the plan of the path of the point *p′p* situated on a produced diameter of the base of the cone B, as the latter makes one complete revolution.**

Draw a plan of the two cones; also draw a plan of the path of P if the rolling cone simply rotated about a fixed axis OR. The plan of the path of P, with this assumption, is the ellipse *pdnf*. But as one cone rolls on the other, divide off the peripheries of the bases of the cones into equal divisions. In plan these equal divisions on the base of the rolling cone are *qs*, *sv*, etc., and the corresponding divisions on the fixed cone are *qi*, *ig*, *gw*, *wk*, etc. As one cone rolls on the other, *s* will come in contact with *i*, and *v* with *g*, and so on. Through *i* and *g* have been drawn lines radiating from O (shown dotted). These have to be drawn through all the other points, *w*, *k*, etc. When the rolling cone has turned through the true angle *qrs*, the point *s* will be in contact with *i*, and the line *sd* will coincide with *ia*. The line *qp* will have moved further over still, and the point *p* will have moved as far on the other side of *ia* as *d* is on the lower side of *pq*; in other words, the new position of *p* relative to *ia* will be the same as the position of *d* relative to *qp*, but on the other side of it. With centre O and radius O*p*, draw the dotted circle *pabe*WV. With *a* as centre and *pd* as radius describe an arc, and with O as centre and O*d* as radius describe another arc, intersecting the former at the point 2. This is a point on the locus. In the same way, when the cone has rolled such that *v* comes into contact with *g*, then *p* will occupy the same position relative to *gb* that *n* does to *qp*, but on the other side of it. Hence with *b* as centre and *pn* as radius, describe an arc; also with O as centre and O*n* as radius describe another arc, intersecting the former in the point 3. In the same way, with *e* as centre and *ph* as radius describe an arc with O as centre and O*h* as radius describe an arc; these



arcs will intersect in 6. In this way the curve p2 3 4 5 6, etc., is described.

o. 54, 1893. **A twisted surface is generated by a right line moving on the directors AA′, BB′, CC′; *abca′b′c′* is one position of the generator. Through the point *mm′* on that generator draw a tangent plane to the twisted surface.**

Let the directors AA′, BB′, CC′, the generator *abc,a′b′c′*, the point *mm′*, and the ground-line XY be given.

The tangent plane to the twisted surface at point *mm'* will be the plane containing the generator *abc,a'b'c'*, and the tangent line to the surface at that point. Before this tangent line can be found, we must find a few more of the generators in different positions (in this case two).

To do this, select any number of points on any of the directors, such as points 2 2', 3 3' on director BB'. Then determine, in the usual way,* a plane to contain the director A and passing through point 2 (say), and also a plane to contain the director C and passing through point 2. Their intersection *v2nq,v'2'n'q'* will be the generator required passing through point 2, and meeting the director CC' in point *qq'*.

Proceed similarly to find the generator passing through point 3 3' viz. *3hg,3'h'g'*.

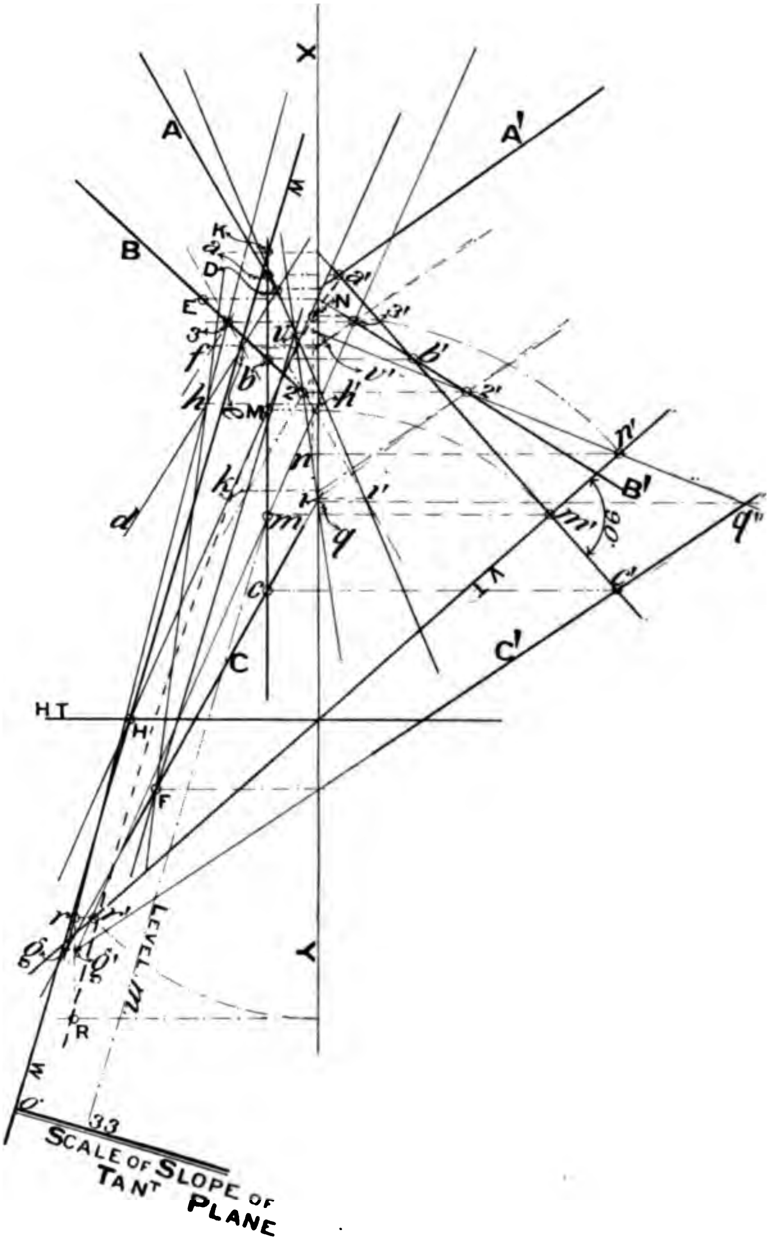
Next cut the surface by a plane perpendicular to the generator *abc,a'b'c'*, and passing through point *mm'* (shown by its traces HT and VT), cutting the new generators in point *nr,n'r'*. Rotate this cutting plane about its H.T. to the horizontal plane, carrying the points *rmn, r'm'n'* with it to RMN.

Through these points draw the curved dotted line as shown, which is the outline of the section made by the cutting plane of the twisted surface generated by the lines *abc,a'b'c'*, *v2nq,v'2'n'q'*, and *3hg,3'h'g'*, rotated into the H.P. Draw the tangent line MH to the surface, intersecting the H.T. of cutting plane in point H. Then through H and K (the H.T. of generator *abc,a'b'c'*) draw the line WW. Then WW is the H.T. of the required tangent plane (to the twisted surface at point *mm'*), shown by its scale of slope.

---

* Through point 3 3' draw a line parallel to the director AA'; its H.T. is at *e*. Through *e* and point D, the H.T. of AA', draw the line D*d*; then D*d* is the H.T. of the plane containing the director AA' and passing through point 3 3'. Similarly, draw through 3 3' a line parallel to the director CC'; its H.T. is at *f*. Through *f* and point F, the H.T. of director CC', draw the line F*f*; then F*f* is the H.T. of the plane containing director CC' and passing through point 3 3'. These traces intersect in point *h*. Join *h*3 and produce it to intersect the director CC' in point *gg'*. Then *h3gh'3'g'* is the intersection of the two planes just found, and is also a generator of the twisted surface.

In the same way, find the generator *v2nqv'2'n'q'*.


X
A
A'
B
K
a
D
E
3
f
b
h
M
w
a'
N
3'
b'
v'
2
h'
2'
n'
n
k
B'
r
r'
m
m'
q''
d
q
90°
c
c'
C
C'
HT
H'
F
LEVEL
m
Y
r
r'
Q
Q'
15
R
o
33
SCALE OF SLOPE OF TANT PLANE

CHAPTER VI.

ISOMETRIC PROJECTION.

THE projection of a cube on a plane perpendicular to one of its diagonals will be a true hexagon, with six lines radiating from the centre point, three of which represent the upper solid angle, while the other three represent the lower angle of the cube.

This bounding hexagon, together with the six radiating lines, are termed the "isometric projection" of the cube.

The three plane surfaces meeting the vertical diagonal, either in its upper or lower extremity, are equally inclined to the plane of projection.

So, also, are the three lines which are the projections of the intersections of these planes, equally inclined to the plane of projection. It will also be seen that the projections of these three lines, which are mutually perpendicular to one another, equally inclined to the plane of projection, and meeting in the same point, will be inclined to each other at $\frac{360^\circ}{3} = 120^\circ$.

The projected lengths of these lines upon the plane of projection will evidently be foreshortened, and bear a constant ratio, to their original lengths in all isometric projection.

Hence the practical method of drawing the isometric projection of any object is to use the T square and a set square having angles of 30° and 90°. Then draw a line in any convenient position, and from any suitable point in it, as at A (Fig. 1), draw the lines AD, AB at an angle of 30°

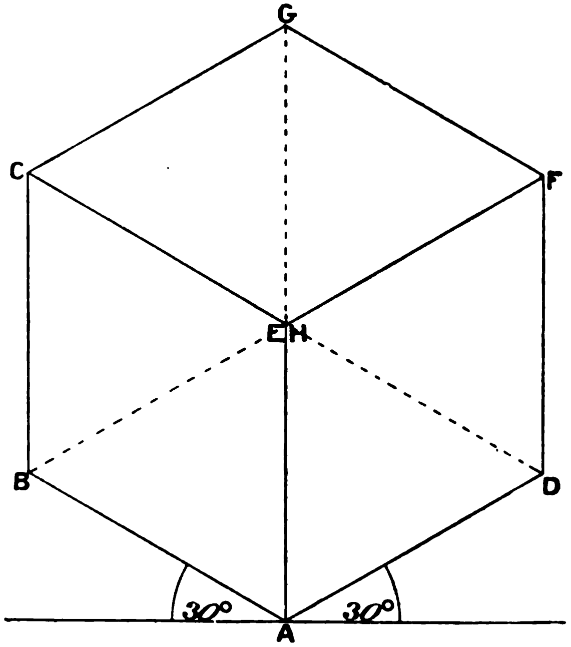


FIG I

to the right and left, and AE perpendicular to the base line respectively.

These three lines are the isometric axes, and on these, or on lines parallel to them, all measurements must be taken.

Next cut off AD, AB, AE equal to the length of the edge of the cube, and by drawing parallels from each point thus found, complete the cube. The drawing thus obtained will represent a cube of slightly larger dimensions than the original one, in the ratio of $\sqrt{3} : \sqrt{2}$ (unless an isometric scale has been used), because the lines are all inclined to the plane of projection, and are not, therefore, seen in their true lengths.

To draw an isometric scale.

Let the three lines AB, AE, AD (Fig. 2) represent the projection of the three lines forming a solid angle of the cube, meeting at either extremity of the diagonal.

Then the true length of the lines may be readily determined, on the principle of orthographic projection. Thus, knowing that the angle BAD is a right angle, join BD, and on it describe the semicircle shown dotted, cutting the line AE in point A′.

Then BA′ is the true length of the line BA, the isometric projection of the edge of the cube.

The following construction may therefore be readily adopted to draw an isometric scale.

Draw a triangle, ABC (Fig. 2*a*), similar to the triangle BAA′ (Fig. 2), of which the angle BAE = 15°, and the angle ACB = 45°.

Now on the side AC mark off the divisions of the original scale, and from each point draw a line parallel to CB to cut the line AB. Then the divisions on AB will be proportional to the divisions on AC, and the original lengths : the isometric lengths : : AC : AB : : $\sqrt{3} : \sqrt{2}$.

For let BC, and CA′ be denoted by 1 (Fig. 2);

Then $BA' = \sqrt{2}$ (Euc. I. 47)

In the $\triangle BCA$, $\angle BAC = 60^\circ$

$\angle CBA = 30^\circ$

$\therefore \triangle BAC = \frac{1}{2}$ equilateral $\triangle$

Let $CA = m$.

Then $BA = 2m$

Let $BC = x$.

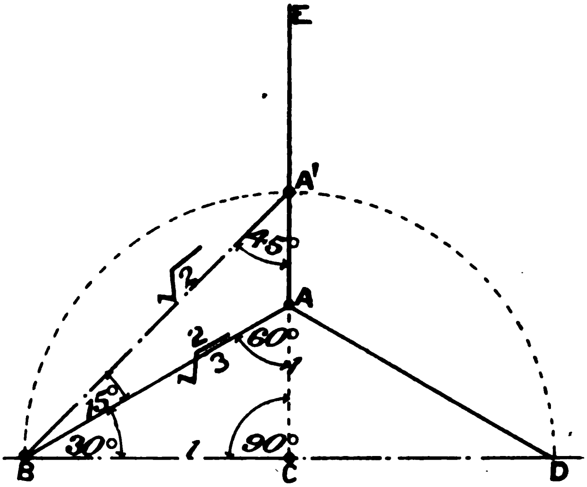

E
A'
45°
√2
A
2/√3
60°
1
15°
30°
1
90°
B
C
D

FIG. 2

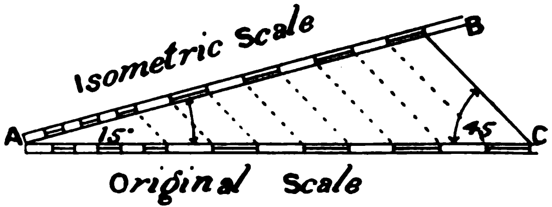

Isometric Scale
B
A
15°
45
C
Original Scale

FIG 2a

$$\text{Then } x^2 = (2m)^2 - m^2 \text{ (Euc. i. 47).}$$

$$\therefore x^2 = 3m^2$$

$$\therefore x = m\sqrt{3}$$

$$\text{But } \frac{\text{BA}}{\text{BC}} = \frac{2m}{m\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$\therefore \frac{\text{BA}'}{\text{AB}} = \frac{\sqrt{2} \times \sqrt{3}}{2} = \frac{\sqrt{3}}{\sqrt{2}}$$

Hence the true length : isometric length as the $\sqrt{3} : \sqrt{2}$.

Again, if a circle be drawn in isometric projection, it will be found to be an ellipse, the major axis of which will be equal to the diameter of the circle, while the minor axis will bear a fixed relation to it.

Circumscribe the circle by a square, and draw the isometric projection of both, as at ABCD (Fig. 3). The diameters of the ellipse parallel to the sides AB or AC are called isometric diameters, at *b*F*t*.

It will be seen that the diameter of the original circle is projected in isometric projection into the isometric diameter, and that the semi-major axis FH is exactly equal to the semi-diameter of the circle.

Now, if HG be drawn parallel to the side BD, and cutting the line AFD in point G, then FG will be the semi-minor axis of the ellipse.

In the triangle FGH, the $\angle$ FGH $= 60°$.

$\angle$ FHG $= 30°$.

Let FG be denoted by m

$$\text{Then GH} = 2m$$

$$\text{Therefore FH} = m\sqrt{3}$$

Hence the ratio of the major axis : minor axis : : $\sqrt{3} : 1$.

Draw G*c* parallel to a major axis and equal to FG.

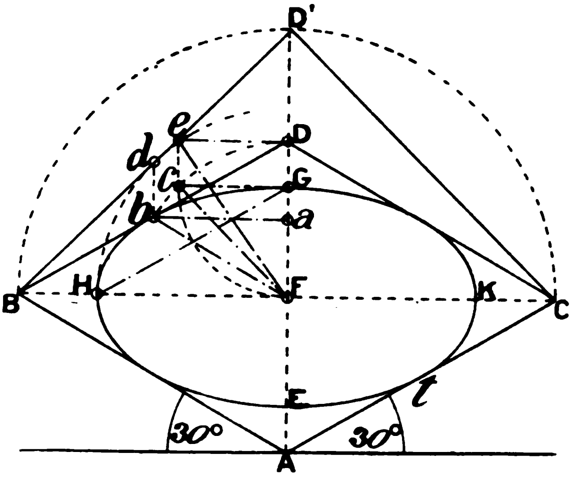


FIG 3

Let FG, Gc be denoted by 1. Then Fc will be $\sqrt{2}$ (Euc. I. 47).

$Fc = FD = Fb$, the isometric diameter of the ellipse. Then the major axis : isometric diameter : minor axis : : $\sqrt{3} : \sqrt{2} : 1$, as before.

Another method of constructing a scale is shown at Fig. 3*a*.

Draw any line, as AB′, of indefinite length; from A measure off a unit of length, as at AC, and erect CB perpendicular to AC = AC = 1.

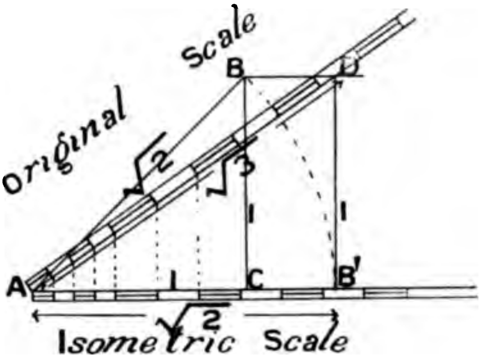


FIG 3a

Join AB; then $AB = \sqrt{2}$.

Now with A as centre, radius AB, draw the arc BB', cutting AC produced in B', and at B' erect B'D perpendicular to $AB' = AC = 1$.

Join AD. Then $AD = \sqrt{3}$.

$$\therefore AB' : AD :: \sqrt{2} : \sqrt{3}$$

On line AD measure off the length of the original scale, and draw lines parallel to DB', cutting the line AB' in points, which give the isometric equivalent to the original lengths.

Draw the isometric projection of the angle bridle joint, shown by its plan and elevation (Fig. 4).

Isometric scale need not be used.

It will be convenient to draw the complete parallelopiped forming the bottom piece, ignoring the bevel end at EB; and measure the amount of slope of the upper piece (seen in elevation), and transfer to A'B' on the isometric drawing. Join EB and produce it till it is 7½ in. long, and complete by drawing parallels, as shown.

As before stated, isometric projection is not suitable for drawings of oblique forms. Thus the drawing just obtained does not give a correct idea of the joint; the length of the bevel line from E to B in the elevation is greatly lengthened at E'B' in the isometric projection.

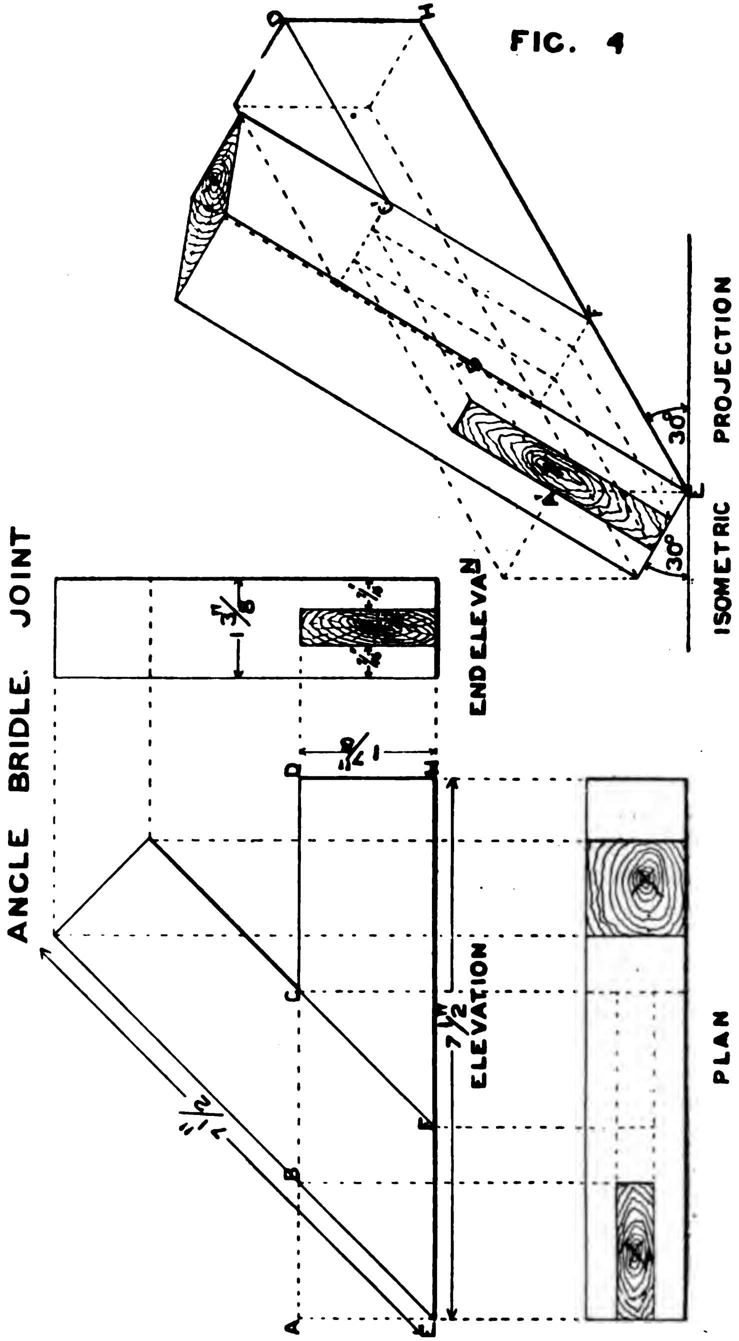


FIG. 4

ANGLE BRIDLE JOINT.

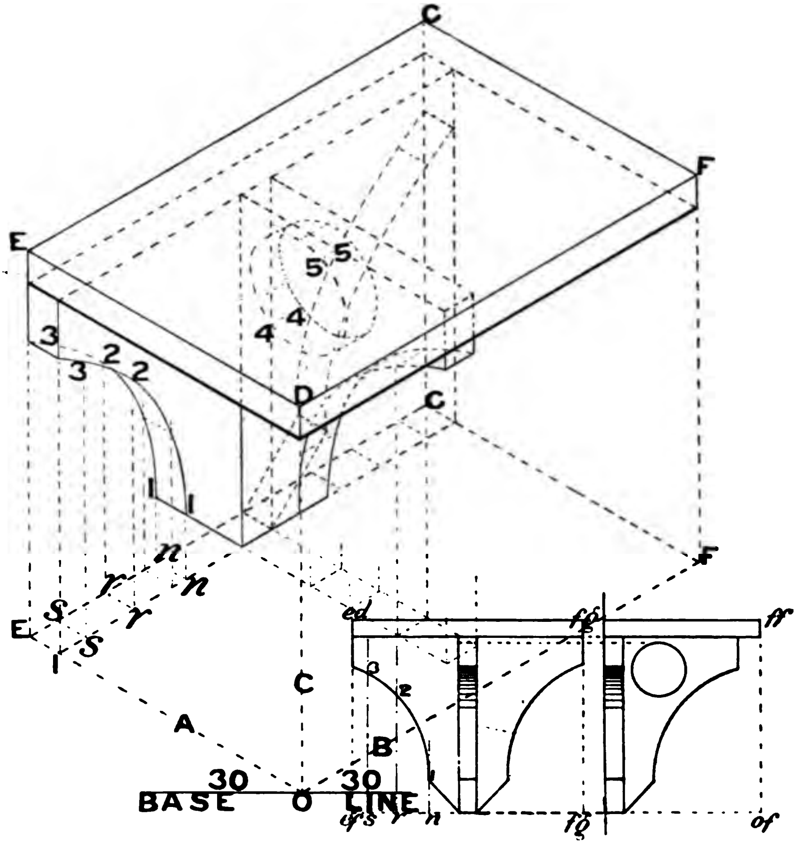
No. 52, 1887. **Make an isometric view of the given bracket NB. An isometric scale is not to be used; dimensions may be taken direct from the figure.**

The front and side elevation (small sketch) being given, the isometric drawing is double the scale of the given figures.

It is found expedient to shroud objects of this kind with rectangular faces, forming a rectangular prism, the prism being put in isometric projection as OEDFG. The lines OA, OB make an angle of 30° with the base-line, and the line OC is perpendicular to it. These lines are called the isometric axes, and all measurements must be taken on them or on lines parallel to them. No detailed description is necessary, as all the lines are shown. Commencing at the top, we get the thickness as shown, the back and the middle piece supporting the shelf in a similar manner. The curves may be readily got by selecting ordinates *n*1, *r*2, *s*3 (see small sketch), and transferring lengths equal to them on the isometric projection, and the curve drawn freehand. Similarly the circular hole may be obtained.*

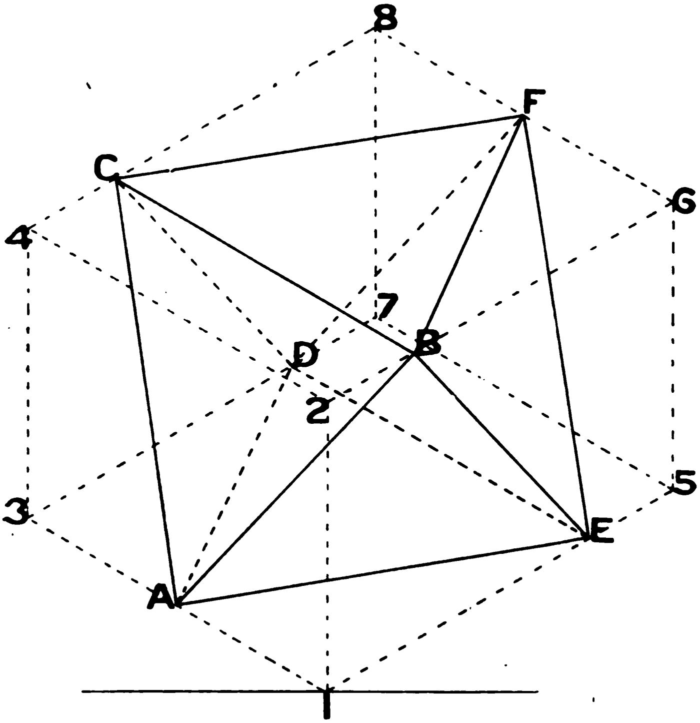
---

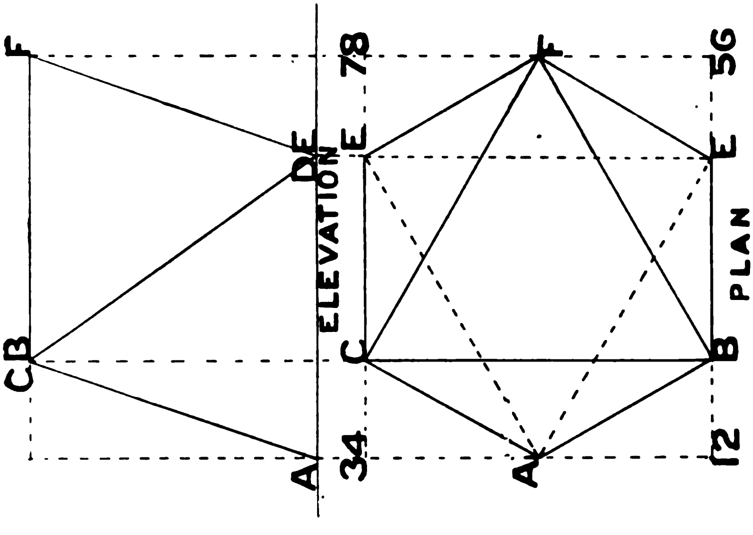
* For text-book description, see Angel's "Geometry and Projection," chaps. xv. xvi.


C
F
E
D
C
F
E
5
5
4
4
3
3
2
2
I
I
n
n
r
r
S
S
I
A
C
B
30
30
BASE O LINE
ed
fe
fr
of
fe
3
2
1
ers r n

No. 49, 1888. **Make an isometric projection of an octahedron of 3·75 in. edge, resting with one face on the horizontal plane. (N.B.—An isometric scale need not be used.)**

Draw the plan and elevation of the octahedron, and enclose it with a rectangular prism as shown at 12, 34, 56, 78. Now draw a base-line, and place the enveloping rectangular prism in isometric projection. Transfer the distance of each corner of the solid away from the corner of the prism, as B from 2 in the plan to B2 in the isometric projection, E2 in plan to E2 in isometric projection, and so on for every corner. These being joined in order, the octahedron is completed.


8
F
C
G
4
7
D
B
2
5
3
E
A
1


F
DE
CB
A
ELEVATION
78
E
C
34
F
A
5G
E
PLAN
B
12

No. 62, 1892. **A section and plan of an open box are given. Show in isometric projection a sectional elevation of the box on *ab*.**

The plan and section (Fig. 1) are given.

The isometric projection of the box may be drawn in various positions. It may have either of its sides in the isometric planes or axes; or, as in the drawing, with plane of section in the isometric plane, making 30° to the left. It will be advisable, as before, to shroud the box with rectangular plane surfaces perpendicular to each other, shown in plan by the rectangle 1 6 8 9. Commence in the usual way by drawing the isometric axes. Now take the distances 1 2 3 . . . 7, Fig. 1, and plot off these measurements on the isometric axes as shown, and complete the rectangle. Then measure off distances to every angle of the box from the edges of the rectangle, also their heights above XY, taking care to always measure on the isometric axes, or on lines parallel to them. No further description of this is necessary, as the solution is exactly similar to the preceding problems. The student is recommended to draw this question to a large size.

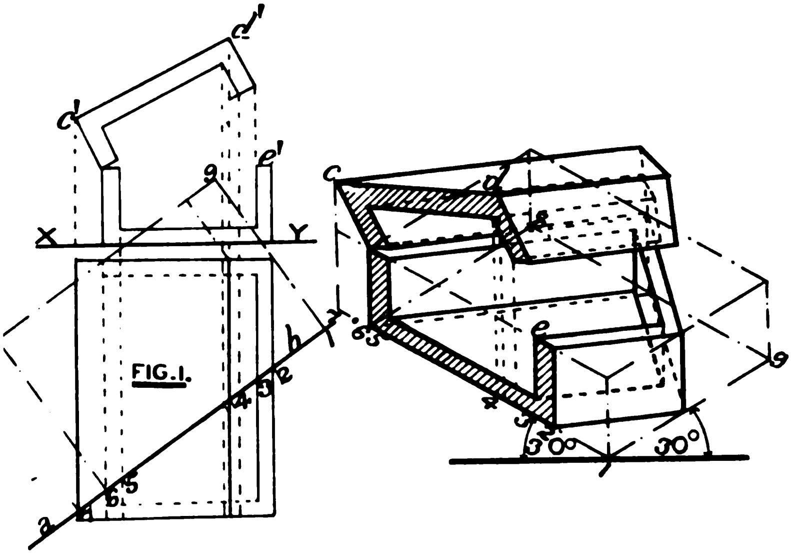
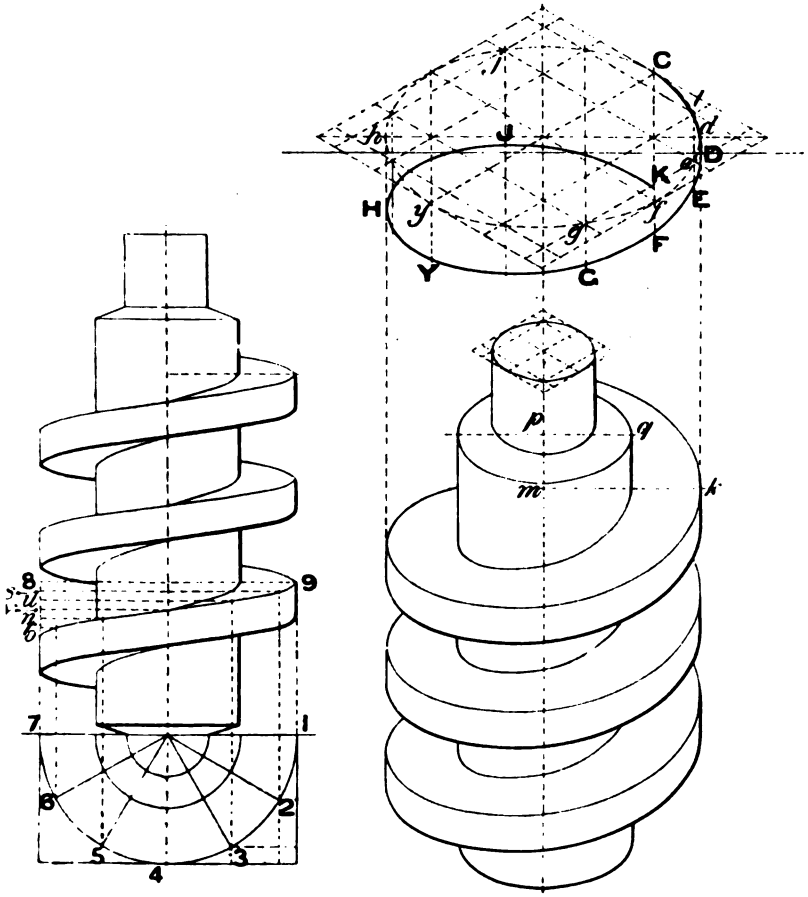


FIG. I.

No. 62, 1893. **Draw the screw in isometric projection, the axis to be represented as vertical, and the nearest side that which is nearest to the observer in the given elevation.**

The elevation of the screw is given on the left of the figure, and a half-plan of the same has been drawn below it, with the larger semicircle shrouded with half a square. This large circle has been drawn in isometric projection at *hjCdefg*, with *hd* as major axis, the ordinary method of placing a circle in isometric projection being used. The vertical isometric axis contains the axis of the screw. The isometric projection of the point 1 is at C; similarly for 2 at *t*, and 3 at *e*; also 7 at *y*. Take the elevation of 1 at 9. The elevation of 2 is at a level below 9 a distance 8*s*. Similarly, the elevation of 3 is at a level 8*u* below 9, and so on for all the points. As vertical heights in the isometric elevation are the same as those in the ordinary elevation, the isometric screw curve can now be drawn. Starting with the thread at C, set off below *t* a distance equal to 8*s*, and below *e* a distance E*e* = 8*u*, giving the point E; further, the distance *f*F = 8*v*, and *g*G = 8*n*, until *k* is reached. On joining up these points a complete convolution is represented in isometric projection.

In the elevation the point 9 is on the extreme right of the figure: but the isometric projection of 9 is at C; *i.e.* the placing in isometric projection turns the body through 45°, and as the same face in isometric projection must appear as in the given elevation, the screw must be turned backwards through 45° to bring C round to the extreme right at D. In this part of a turn, the whole will be lowered through one-eighth of the pitch; therefore set down *d*D = $\frac{1}{8}$ pitch, and draw the horizontal line through D shown full. The point 9 is now at D, and as the isometric curve is the same wherever situated or how much the screw is turned, the curve CDGHK can be reproduced as often as necessary by simply shifting it bodily parallel to itself along the vertical axis. Thus D has been moved to *k*, and every



point in the curve through *k* is at the distance D*k* from the curve CDG, etc. The inside curve at the root of the thread can be produced in the same way. The complete isometric projection is given. If an isometric scale were given, the construction would be the same, but all dimensions would be reduced by means of an isometric scale.

A quicker method would be to arrange the shrouding square so that one of its diagonals be perpendicular to the V.P., *i.e.* passing through point 4 and the axis of the screw, when the isometric projection may be drawn direct.

CHAPTER VII.

SHADOWS.

FOR all purposes in geometry, we may consider that rays of light emanating from any source whatever always travel in straight lines, although this is not rigorously true, except where the medium through which the light travels is homogeneous.

Now, if an opaque body be placed in the path of these rays, those which fall on the body will be intercepted; consequently the surface upon which these rays impinge will be illuminated, while the remainder will be in shade. The rays which are not intercepted by the body, but tangential to it, continue their course in straight lines, forming the surface of a prism, cylinder, pyramid, or cone, depending upon the outline of the body whose shadow is required, and on the source of light. Thus, if the solid is contained by planes, and the rays of light considered parallel, then a prism is formed; while if the body is contained by curved surfaces, a cylinder will be formed.

If the rays be considered as diverging from a point, then in the first case a pyramid is formed, and in the second a cone, the vertex of which is the source of light.

If the contact points of the surrounding rays with the body be joined, a definite *outline* is obtained, generally spoken of as the line of demarcation, or shade line, *i.e.* the line that separates the illuminated from the unilluminated portion of the object (see Fig. 5).

Imagine every point on this outline to be connected to the luminous point by straight lines, generating the surface enveloping the object, and supposed indefinitely produced.

The intersection of this surface with the "planes or surfaces" receiving the shadow will be the outline of the shadow required, or, in other words, the shadow of any point is the *intersection* of the line joining that point to the source of light, with the receiving surface under consideration.

1. **A tetrahedron is given by its plan and elevation; determine the shadow cast on both planes of projection.**

The rays of light are parallel, and their direction is given in plan and elevation by the arrows *r*, *r′*.

Through *bb′* draw the projection of a ray of light, and determine its horizontal trace as at B; similarly, D is the horizontal trace of a ray through *dd′*. Then BD is the shadow of the edge *bd* cast on the horizontal plane. Again the shadow of the edge *bc* or *dc* will be found by determining the shadow of the point *c*, as at C′, and joining C′B or C′D. The projection of a ray through *c* will be found to pierce the vertical plane at point C′, before reaching the horizontal plane. Now, as the shadow of the extremity *b* is at B on the H.P., and that of *c* at C′ on the V.P., these extremities must not be joined, because they are contained by different planes; it will therefore be necessary to determine its direction on one plane, preferably the H.P., supposing the V.P. removed for the time being; then its crossing-point with *xy* may be joined to C′. The H.T. of the ray through *c* will be at C, and BCDC would be the complete shadow on the H.P. These lines cut *xy* in points E and F, the portion BEDF in front of the V.P. will be shadow on the H.P., and EC′FC′ the shadow on the V.P. Then BDBEC′DFC′ is the entire shadow of the triangular base *fcd* of the tetrahedron on both planes of projection.

Now determine the shadow of the vertex *aa′* as at A′ in precisely the same way, and join to each corner of the base: the contour-lines of all the points found will be the shadow required.

Note.—However, we may consider the prism of rays enveloping the object and determine its intersection with both co-ordinate planes; its complete intersection with the H.P. (supposing the V.P. removed) is BDAC, and with the V.P. (supposing the H.P. removed) B′D′A′C′, the two outer lines crossing the *xy* at E and F as before, and the complete outline of shadow is BEA′C′FD.

Prob. 2. **A cone is given by its plan and elevation. Determine the shadow cast on both planes of projection, and the parts in shade on the cone itself.**

The rays of light are parallel, and their direction is given in plan and elevation by the arrows r, r'.

The shadow cast by the cone may be found in precisely the same way as shown in the last problem, *i.e.* by taking a number of points and passing a line through each, etc. But in this case it will be found convenient to determine the line of demarcation, as this is the line that casts the shadow; to do this we must determine two planes tangential to the cone and parallel to the given ray. Thus through cc', the vertex, draw a line parallel to the rays of light, cutting the plane of the base of cone in mm'. Now through m draw two tangents to the circular base of the cone, the contact points of which are nn', pp'; join nc, $n'c'$, pc, $p'c'$. Then these lines are the lines of demarcation on the surface of the cone, and the arc included by $nf'e'p'$ shown in thick line is the line of demarcation of the base.

The complete lines of demarcation, therefore, are the two generators $c'n'$, $c'p'$, and the circular arc $n'f'e'p'$ connecting them at n' and p' respectively.

Now select any number of points on this line, as *fenp*, and determine their shadows, as at FENP, and so on till a sufficient number of points are found. No further description is considered necessary, as all the construction lines are shown.

NOTE.—(1) The line of demarcation is always continuous.

(2) The shadow cast by any plane figure on a plane parallel to itself is an equal figure where the rays of light are parallel. Thus the shadow of the circular base of the cone on the V.P. (supposing the H.P. removed) is the circle shown in dotted lines.

(3) When the rays of light diverge from a point, the shadow cast will be a similar figure, but larger, in proportion to the distance the figure and plane are from the luminous point.

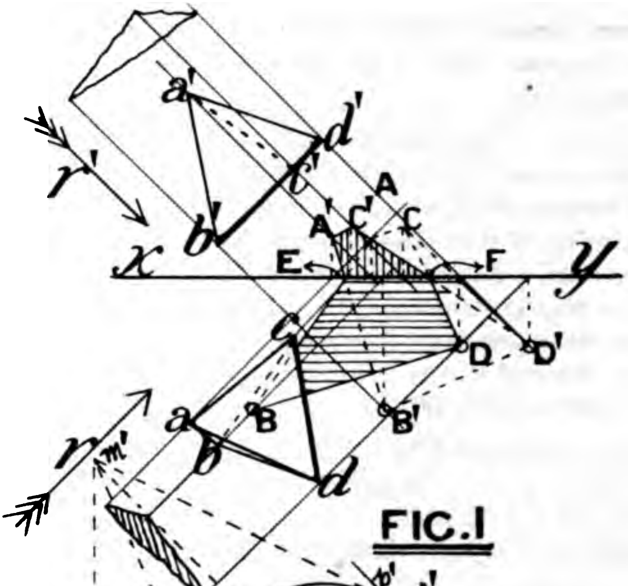

a'
d'
c'
r'
b'
A
A'
C'
C
x
E
F
y
c
D
D'
a
B
B'
r
m'
b
d

FIG. I

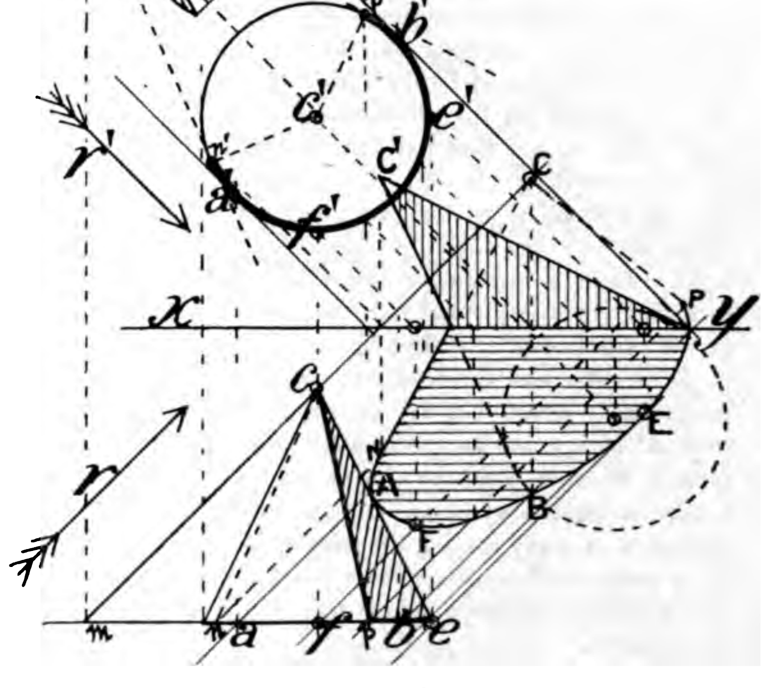

p'
h'
c'
e'
r'
n'
a'
C'
f'
C
P
x
y
c
E
N
A
B
r
F
m
n
a
f
p
b
e

FIG. 2

Prob. 3. **A square pyramid is given by its plan and elevation. Determine the parts in shade, and the shadow cast on both planes of projection.**

Here the rays of light are parallel to a pair of the base edges of the pyramid in elevation, and it will be seen that the plane surfaces *a'e'c'*, *a'e'b'*, *b'e'd'* are illuminated, also that the plane *c'e'd'* is in shade. Then the line of separation in elevation is *e'd'c'*, and in plan *abde*. Now proceed in the usual way to determine the shadow of each of the angular points separately. The contour-lines joining the points thus obtained will be the outline of the cast shadow required, shown at ECA'B'DF.

Prob. 4. **A cylinder, surmounted by a horizontal slab having four circular edges, is given by its plan and elevation. Determine the shadow cast on both planes of projection and on the solid itself. Direction of rays are shown by the arrows *r*, *r'*.**

Commence by determining the lines of demarcation on the cylinder by drawing two tangent planes to it, and parallel to the rays of light; then the surface included by these lines and on the illuminated side of the cylinder will be the surface that will receive the shadow that may be cast upon it by the top. This part of the shadow may be readily found, if we imagine a series of vertical sections taken through the solid by planes parallel to the rays of light. Thus a section of the solid made through the corner *ab* (in plan) would cut the surface of the cylinder on the near side in a vertical line.

Draw this line in elevation, and through *a'* draw the projection of a ray of light; cutting this section line in A'; then A' is a point in the shadow cast on the cylinder. It will be seen that the lower edge of the slab casts a shadow on the cylinder. Again, through *cc'* draw the projection of a ray cutting the section line in C', and so on for a number of points. These points, being joined, will form a continuous curve, which will be the shadow cast on the cylinder.

The shadow cast on the H.P.* and V.P. is found in precisely the same way as described in the previous problems.

---

* See note, Prob. 1.

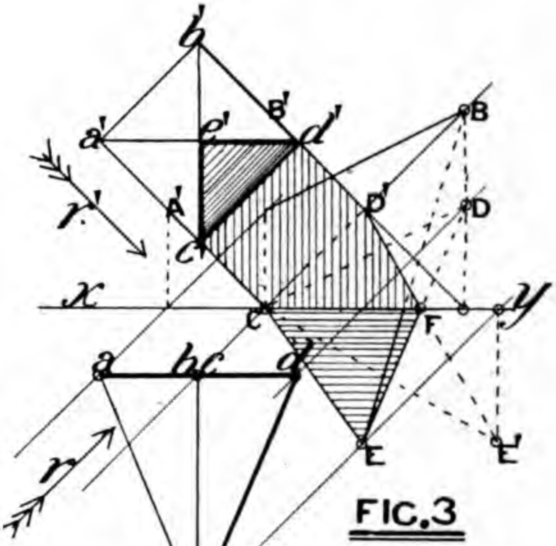

b'
B'
a'
e'
d'
B
A'
D'
D
r'
c'
x
y
c
F
a
b c
d
E
E'
r

FIG. 3

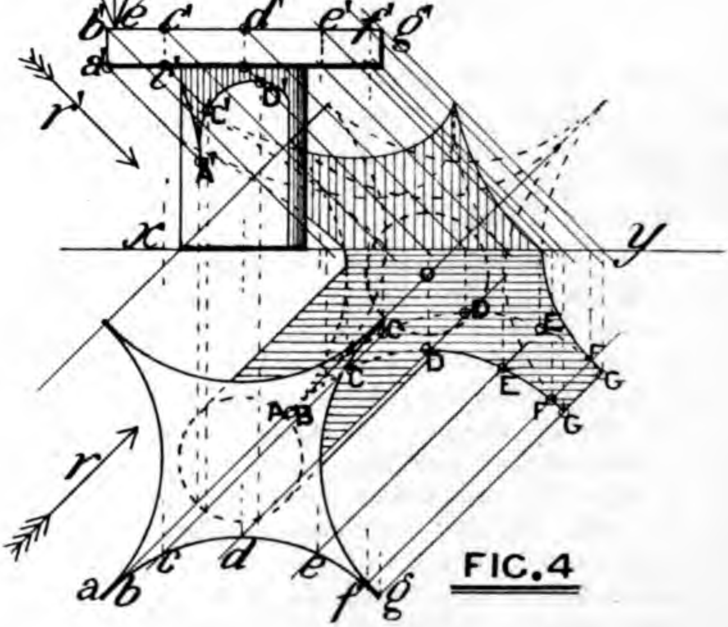

b' e' c' d' e' f' g'
a'
C'
D'
r'
A'
x
y
D
AC
C
D
E
F
G
F
G
A B
r
a b c d e f g

FIG. 4

Prob. 5. **A sphere is given by its plan and elevation. Determine the shadow cast on both planes of projection; also the parts of the sphere in shade. The direction of the rays of light are shown by the arrows r, r'.**

The student will readily perceive, from the reasoning in the preceding problems, that the rays of light tangential to the sphere will generate a cylindrical surface; consequently the shadow on the H.P. and V.P. will be part of ellipses respectively, and the line of demarcation will of course be the circle of contact of the cylinder with the sphere.

The line of demarcation will be readily found, if a new elevation of the sphere and of the cylinder of rays is made on a plane parallel to the plan of the given ray, as at x_1y_1. The circle of contact is shown here by the straight line $a''d''$; its plan is the ellipse $abcd$, and its elevation on the original sphere is $a'b'c'd'$. It is obvious that the shadow on both planes of projection may be found by taking a series of points in the line of demarcation, as explained in the previous problems. (See note, Problem 1.)

Prob. 6. **A sphere is given by its plan and elevation. The rays of light emanate from the luminous point ll'. Determine the shadow cast on both planes of projection, and the parts of the sphere in shade.**

Take a new ground-line $x'y'$ parallel to the plan of ray from l, and passing through the centre of the sphere, *i.e.* parallel to the axis of the *enveloping* cone, and make a new elevation of the sphere and its enveloping cone on x_1y_1. Determine the circle of contact, which is shown by the straight line $a''b''$. Complete its plan $acbd$, and its elevation $a'b'c'd'$ on xy; this circle is the line of demarcation on the sphere, as in the previous problem, and is of course the line which casts the shadow; the part of the sphere's surface remote from l is in shade as shown. The shadow on the H.P. is the part of the elliptic trace of this enveloping cone with the H.P. in front of V.P., determined from the new elevation on $x'y'$; thus, draw the axis $l''c''d''$ of the cone, and next bisect the distance between the points where the outside generators meet $x'y'$, in the point F

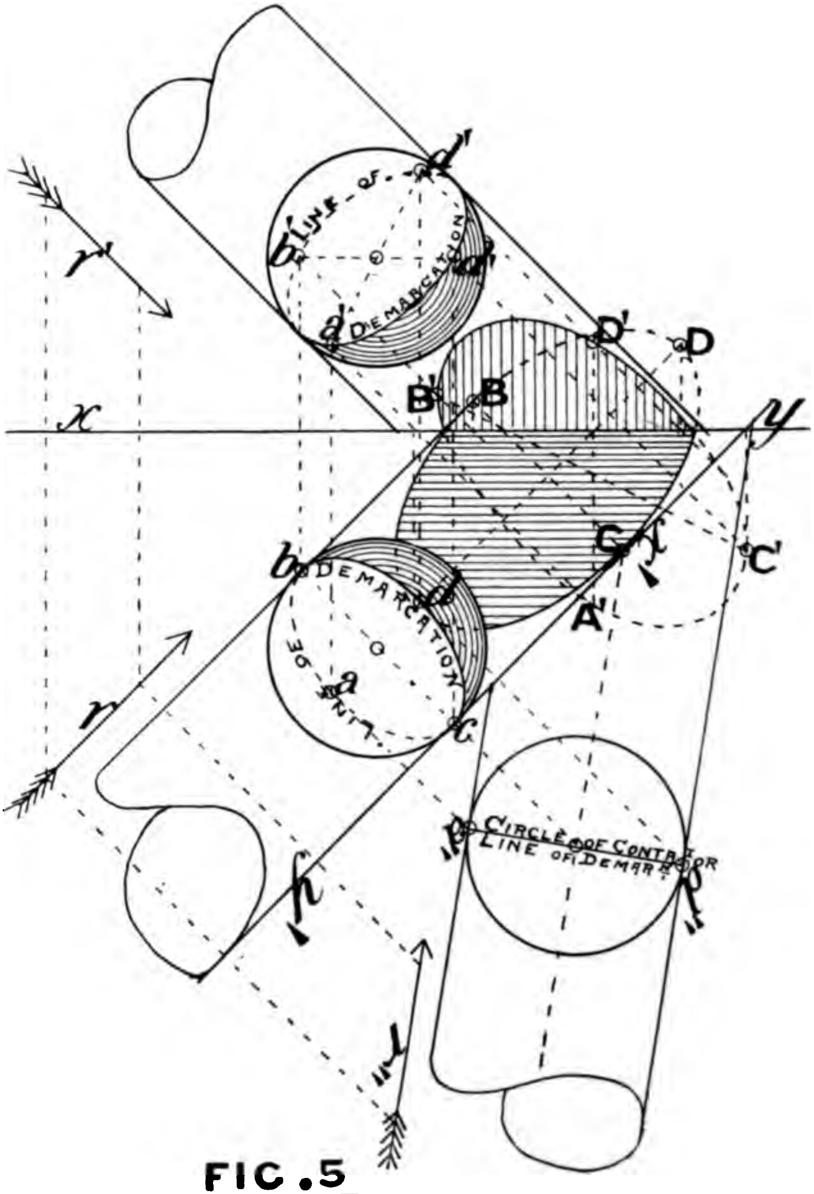

d'
b'
LINE OF DEMARCATION
a'
d'
D'
D
B'
B
x
y
r'
b
DEMARCATION
d
LINE OF
C
C'
A'
a
c
r
CIRCLE OF CONTACT OR
LINE OF DEMARC.
p
p'
h
r''

FIG. 5

Now through F draw a line perpendicular to the axis of the cone, cutting the axis produced in E, and the outside generator in C. With E as centre, and radius EC, describe the arc CH, and draw FH parallel to the axis of cone, cutting the arc in H; then FH is the length of the minor axis of the elliptic trace of the cone on the H.P. The elliptic trace of the cone on the V.P. is not drawn in full, only the portion above *xy*, and is found similarly to that in the last problem.

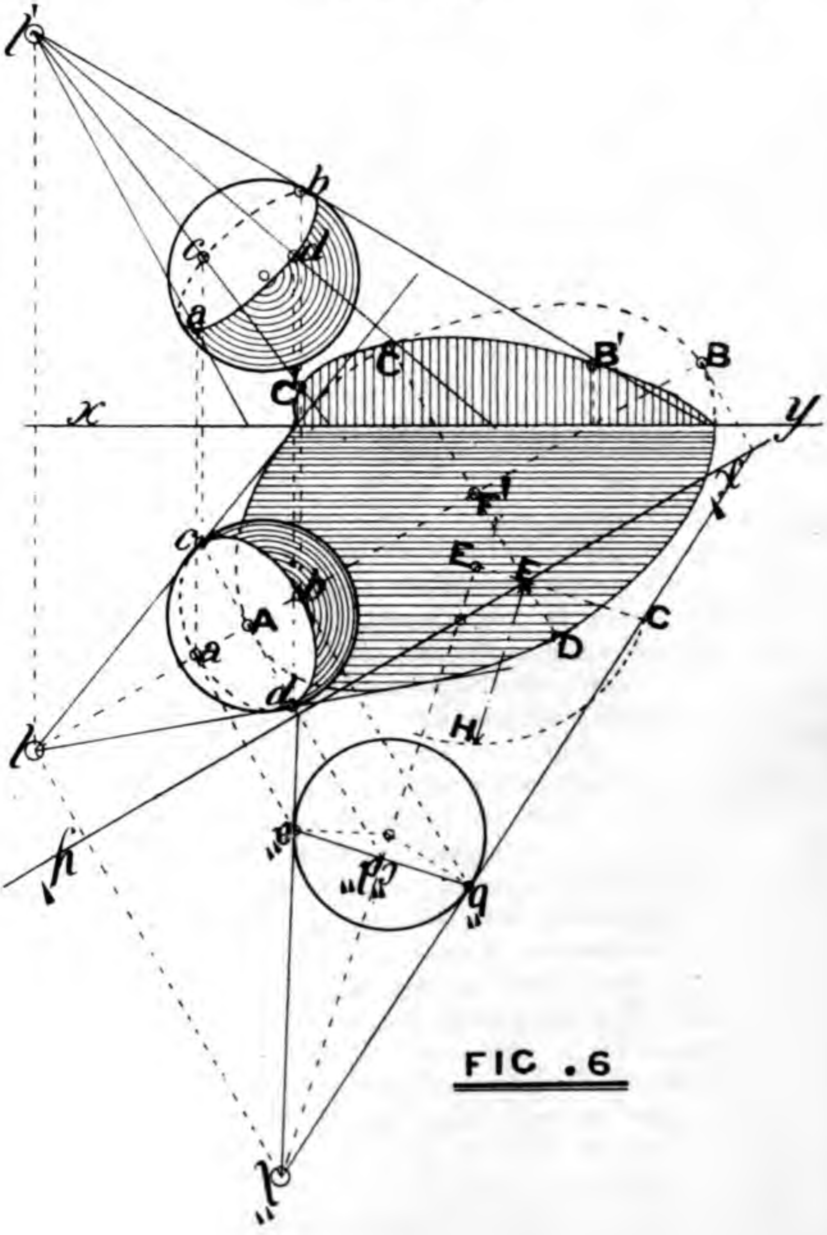
Prob. 7. A circular annulus or anchor ring is given by its plan and elevation. Determine the line of demarcation on the surface of the ring. The rays of light are taken parallel to the arrows r, r'.

Take a ground-line x_2y_2 parallel to the plan of the ray r, and make a new elevation of the annulus and of the ray r on this new ground-line, as at r'A. Now imagine the ring to be cut up into sections by vertical planes* shown by their traces AA, BB, B'B', CC, C'C', etc., and consider the planes AA, DD as the chief planes, AA being parallel and DD perpendicular to the plan of the rays of light, and for convenience take the planes BB, B'B', CC, C'C' at equal angles with plane AA.

Let us consider the section made by the plane AA, parallel to the plan of the ray of light. Draw lines parallel to r'A, tangential to this section at the four points a', a on the outside, and a', a on the inside; then these are the shadow points on this section. Project these points back to the plan and to the given elevation. In plane DD the ray will be tangential to the ring at its centre point d, projected to the plan at dd, and to the given elevation at $d'd'$. Again, let us consider the section made by the plane B'B', which is shown on the given elevation on xy, and the ray r'B. Draw the projections of the rays of light, as before, tangential to the circular sections at the four points, two on the outside at b', b', and two on the inside at b', b'; project these to the plan at b_1b_3,bb: these will be four points on the shadow. Clearly, as the plane BB has

---

* The planes of section must always contain the *normals* to the surface.



FIC .6

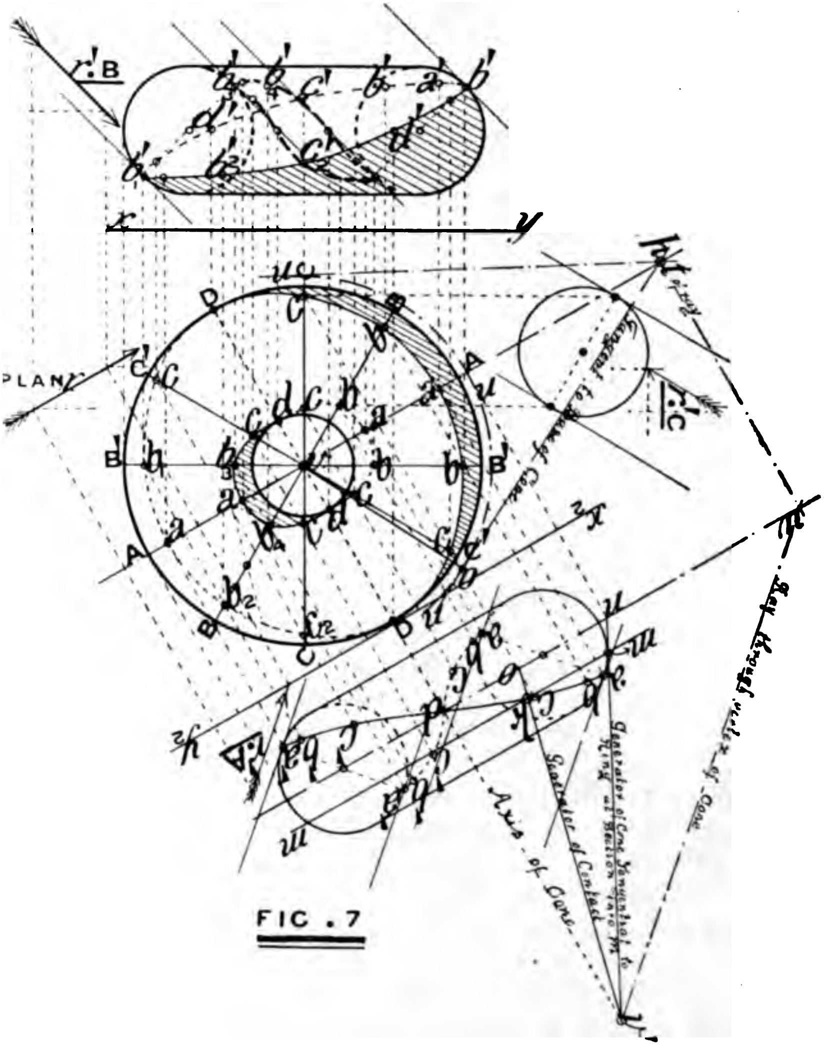
been taken at the same angle with plane AA, as B′B′, four corresponding points may be immediately found in the plane BB, as at b_2b_4 and bb on the other side. Of course, these four points in plane BB are precisely the same height as their corresponding points in the plane B′B′. Similarly consider the section made by the other plane CC, shown at r'C. Draw the tangents to the circle, and project back to the plan and to the given elevation at $ccoo$, and $c'c'c'c'$. Of course, the four corresponding points in plane C′C′ are taken as previously explained.

Now, the curve drawn through these points will be the lines of demarcation; it will be a continuous curve, but not an elliptic curve, as some authors have supposed. The idea that the line of demarcation on the elevation taken parallel to the ray of light to be straight is *erroneous*, and the student is recommended to work out one example to a large size, and prove for himself that the line of demarcation is not straight, as stated by certain authors, but an irregular curve, as shown on this new elevation on x_2y_2, at $a'b'c'dkcba$ of the outside, and $a'b'c'dcba$ of the inside.

Another method is also added as a geometric gymnastic for the energetic student, whereby a horizontal section is taken of the annulus at mm (on x_2 y_2), and a cone is taken tangential to the annulus at this section; thus, join m to the centre of section, and draw the generator of cone perpendicular and tangential to it at m, meeting the imaginary axis in point v'. Suppose we take our horizontal* plane level with the centre of the annulus, and produce the cone till it meets it, having its plan shown partly at uuu. Now through the vertex v' draw the projection of a line parallel to a ray of light, and find its intersection, or, in other words, its horizontal trace *ht* on that plane, as shown. Now draw through the *ht* of the line tangents to the circular base of the cone, and find the line of contact ov, $o'v'$; this contact line is really the line of demarcation on the cone assumed, and consequently the crossing-point of this line with the horizontal section

---

* The horizontal plane may be taken at pleasure, provided that the base of the cone is contained by it.


r'B
x
y
PLAN
r'C
u
m
k
Tangent to Base of Cone
Ray through vertex of Cone
Axis of Cone
Generator of Contact
Generator of Cone tangential to Ring at Section into M

FIC . 7

mm, at *k*, is a point in the line of demarcation. Of course, a corresponding point will be found on the other side. It will be seen that the point *k* falls exactly on the shade curve line already obtained. If the student takes a number of similar cones, at different levels on the ring, and finds a number of points in precisely the same way, he will have obtained exactly the same result as before. Of course, he will see that it is not necessary to take cones with vertices below the centre of the annulus, for sections considered at equal distance below the centre as they are above the centre, will have their points similarly placed, but below on the other side. However, the student is strongly recommended to adopt the former method, as it is very simple and applicable to all cases where sections of the object are readily obtained, provided, however, that such sections contain the *normals* to the surface.

No. 51, 1893. **Determine the shadow cast by the given bracket on the wall to which it is fixed. The rays of light are parallel, and two elevations, r', r'', of a ray are given.**

The front elevation (Fig. 1) and the side elevation (Fig. 2), with ray $r'r''$, are given. Commence by labelling every corner on the bracket, and from each draw a line parallel to the given ray; their trace on the vertical plane will be the points in shadow on the wall. Thus parallel to r'' draw ray from point A, meeting the wall FG in point a, also from point A (Fig. 1) draw parallel to ray r', meeting a horizontal through a in point a'. Then a' is the shadow of the corner A on the wall, for it is the vertical trace of a line through A, and parallel to the given ray $r'r''$. A ray through B meets the wall in point b, projected horizontally to meet the ray from B (Fig. 1) in point b'; similarly for all other points. The circle will cast an elliptic shadow, therefore draw the rays from the ends of the vertical and horizontal diameters respectively. Only a small piece of this is cast on the wall, the greatest part falling on the bracket. The shadows of the front curves are found by taking a series of vertical planes, cutting the curve in points as N. Rays through these points will determine their shadow on the wall as before, *i.e.* a ray through N will meet the wall at n, and projected across to meet the ray from N (Fig. 1) in n'; then, all the contour-lines being joined, the shadow will be complete. It will be noticed that the shadows of the front curves, on reaching the wall, cross one another.

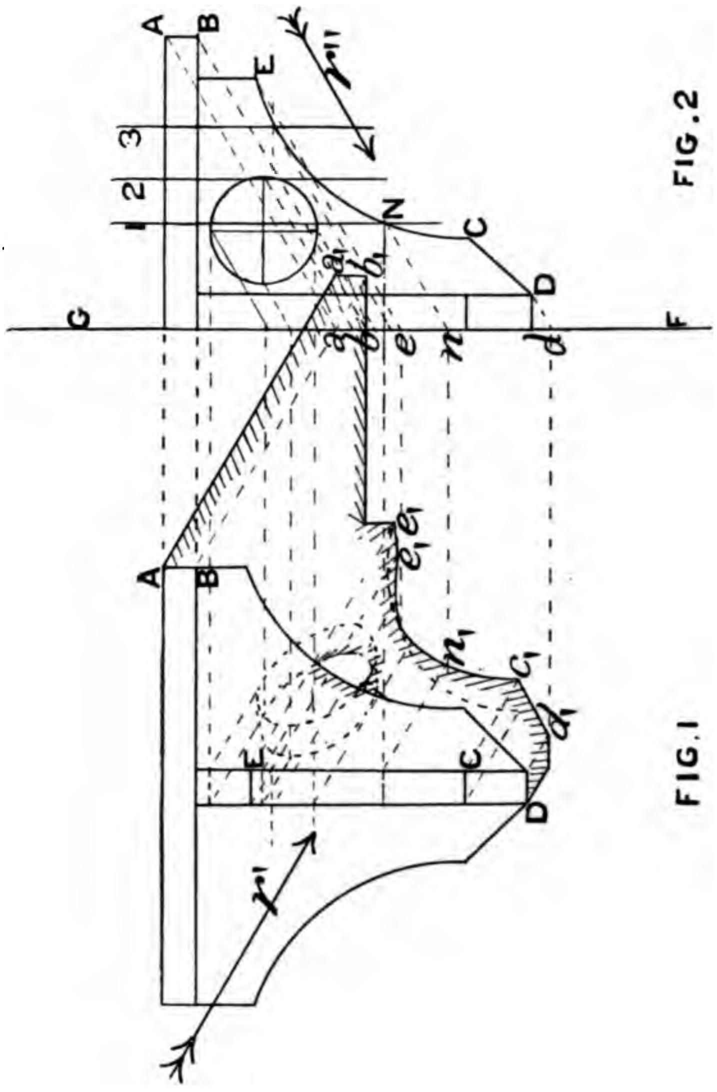
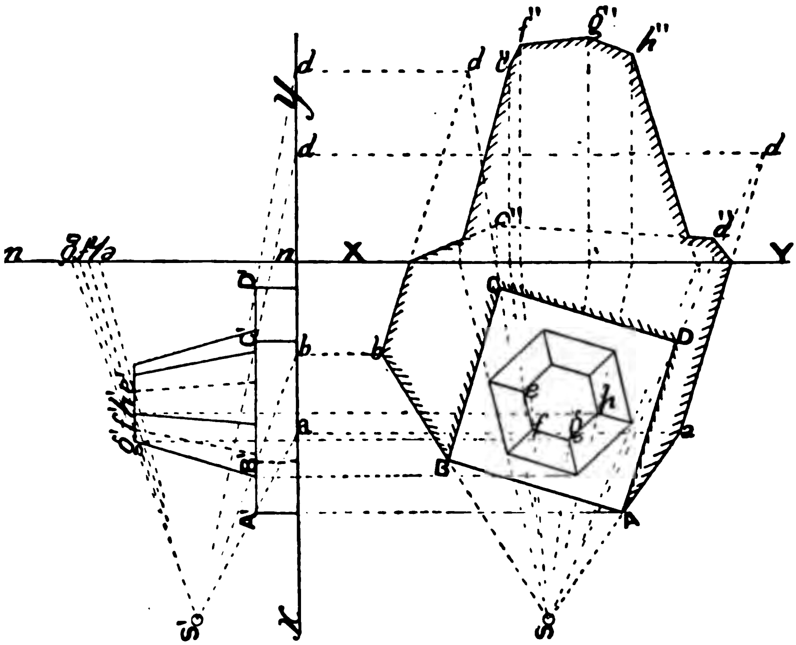


FIG. 2

FIG. 1

No. 50, 1888. **A truncated hexagonal pyramid, height 1·5 in., rests on a square block, height 0·5 in. The plans of both solids are given. Determine the shadow cast upon the horizontal plane and a vertical plane standing on XY. The source of light is a point whose plan is S, and the height above the horizontal plane 1·25 in.**

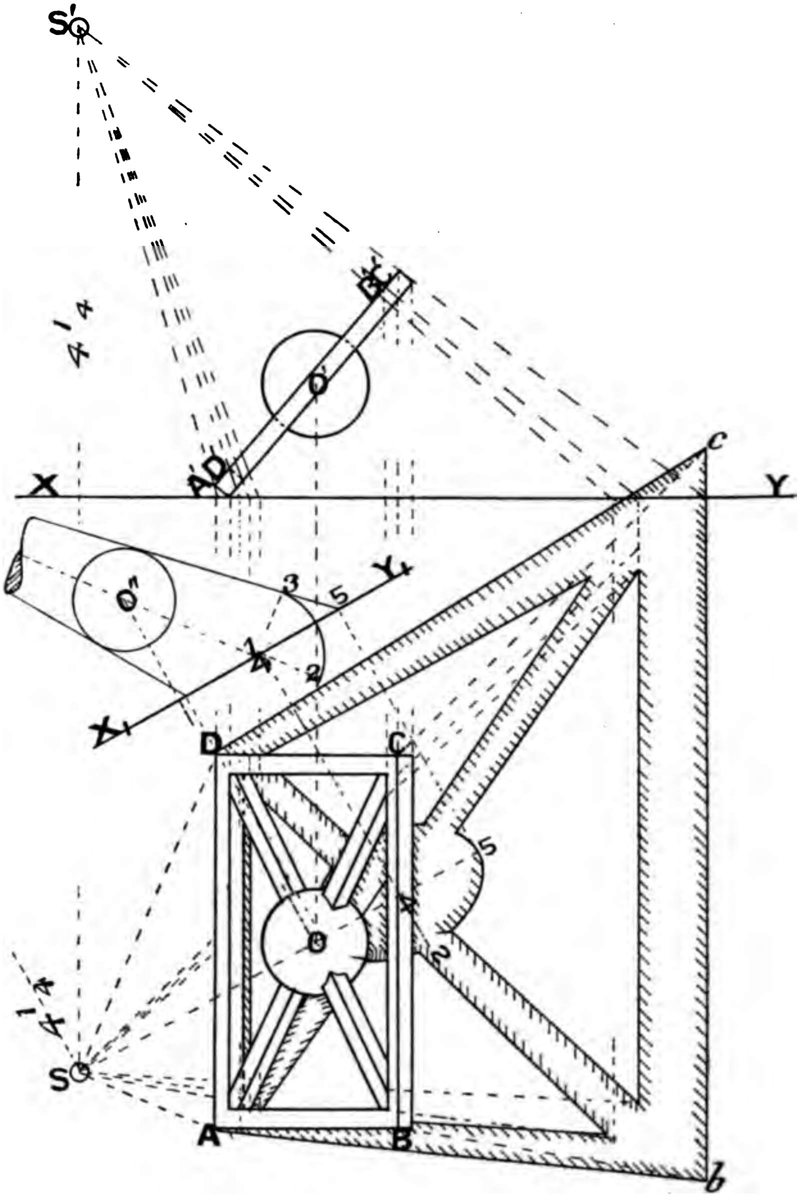
The plan marked ABCD, *efgh*, etc., is given, with the point S. The shadows on the planes are the projections of the lines of demarcation from point S, which are easily found by making a new elevation of the object on a line *xy* perpendicular to the given XY. Obtain S′ 1·25 in. above *xy*, and from point S′S, the source of light, draw a series of rays through every point of the object, and determine their traces on the H.P. and V.P. respectively; thus a ray of light through A meets the ground at *a*, B in *b*, C in *c*, and D in *d*. All these joined will be the outline of the shadow of the top of slab on the ground. But the rays through C and D meet the vertical plane *mn* as shown; therefore take these heights and transfer to the point where the ray in plan cuts *xy*, as in $c''d''$. Join the point where the shadow in plan cuts *xy* as shown. Now do the same with the other points, *efgh* (all the heights are shown on the elevation above *xy*) cutting *mn*, and transfer these heights to their positions, as $e''f''g''h''$. The construction is obvious from the drawing.


f''
g''
h''
d
c'
d
y
d
d
n
n
X
Y
c''
d''
D'
C'
b
b
C
D
e
h
f
g
a
a
B'
B
A
A'
S'o
So
x

No. 52, 1889. **A rectangular panel, with diagonals and sphere at their intersection, is inclined as shown. Determine the shadow cast on the horizontal plane by rays proceeding from a point of which S is the plan, and the height of which above the horizontal plane is $4\frac{1}{4}$ in.**

The given figure is shown in plan by ABCDO, and the elevation by A′B′C′D′O′. The elevation of S is found by the perpendicular to *xy* and $4\frac{1}{4}$ in. above it, as at S′. The shadow cast on the ground is easily obtained by determining the horizontal trace of every ray from SS′, through every convenient point of the object. Thus draw through S, S′C′, SC; its horizontal trace is at *c*; also S′B′, SB will have its horizontal trace at *b*. Then line *cb* will be the shadow of BC on the ground; or, in other words, it is the horizontal trace of a plane passing through S and BC. Such plane will contain all possible rays from S through the edge BC. All other points are similarly found. The inexperienced may find some difficulty in determining the shadow of the sphere on the H.P. Take a new groundline as X_1Y_1 parallel to the ray SO in plan, and make a new elevation of the point S and sphere, centre O″; the rays enveloping the sphere and passing through S″ will form a conical surface, of which its trace on the horizontal plane will be the shadow required. The semi-major axis = 4′5′, semi-minor axis = 4′2′. Always select as far as possible the lines of demarcation, as these only cast the shadow—the circle of contact* of cone with sphere casts the shadow. The line EF casts no shadow, because it is hidden from the illuminating point by the edge BC, and so on. The student will discern several other lines not affected. The contour-lines thus obtained will be shadow required.

* See Fig. No. 6.

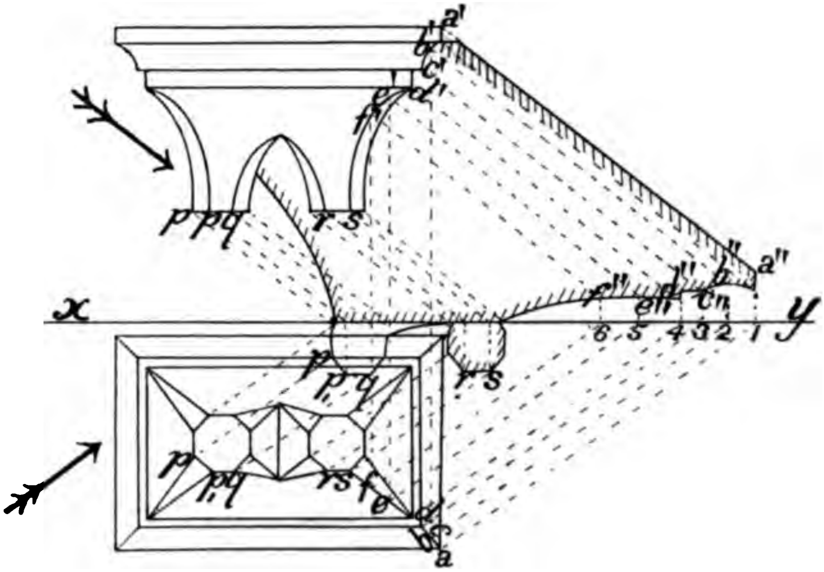

S'
4 1 4
B' C'
O'
A' D'
X
Y
O''
3
5
Y₁
1 4
2
X₁
c
D
C
5
4
O
2
4 1 4
S
A
B
b

No. 62, 1891. **A double capital is given by its plan and elevation. Determine the shadow cast on the horizontal and vertical planes by parallel rays of light, the direction of which is given.**

The shadow of the given object on the horizontal and vertical planes will consist of the contour-line drawn through the traces of all lines which are respectively parallel to the rays of light, and passing through every possible point on the object; thus: through a draw a line parallel to the ray in plan, meeting xy in point 1, and from a' draw a line parallel to the ray of light in elevation, meeting a perpendicular through 1 in point a'', then a'' is the vertical trace of the line through aa' parallel to the ray of light. Proceed in a precisely similar way to find a number of other points, selecting points in the curves, as at b, b'; its trace will be at b''.

Ray through corner cc' will have its trace at c''.
" " " dd' " " " d''.
" " point ee' in the curve will have its trace at e''.
" " " ff' " " " " f''.

It will be noticed that part of the bottom of capital will have their shadow on the horizontal plane, and that only the line of demarcation need be projected, as that only casts the shadow, for the shadow of the inner curves are lost mostly by the shadow of other parts cast on them.


a'
b'
c'
d'
e'
f'
p
p
q
l
r
s
x
y
f''
e''
d''
c''
b''
a''
6
5
4
3
2
1
p
p
q
l
r
s
p
p
q
l
r
s
f
e
d
c
b
a

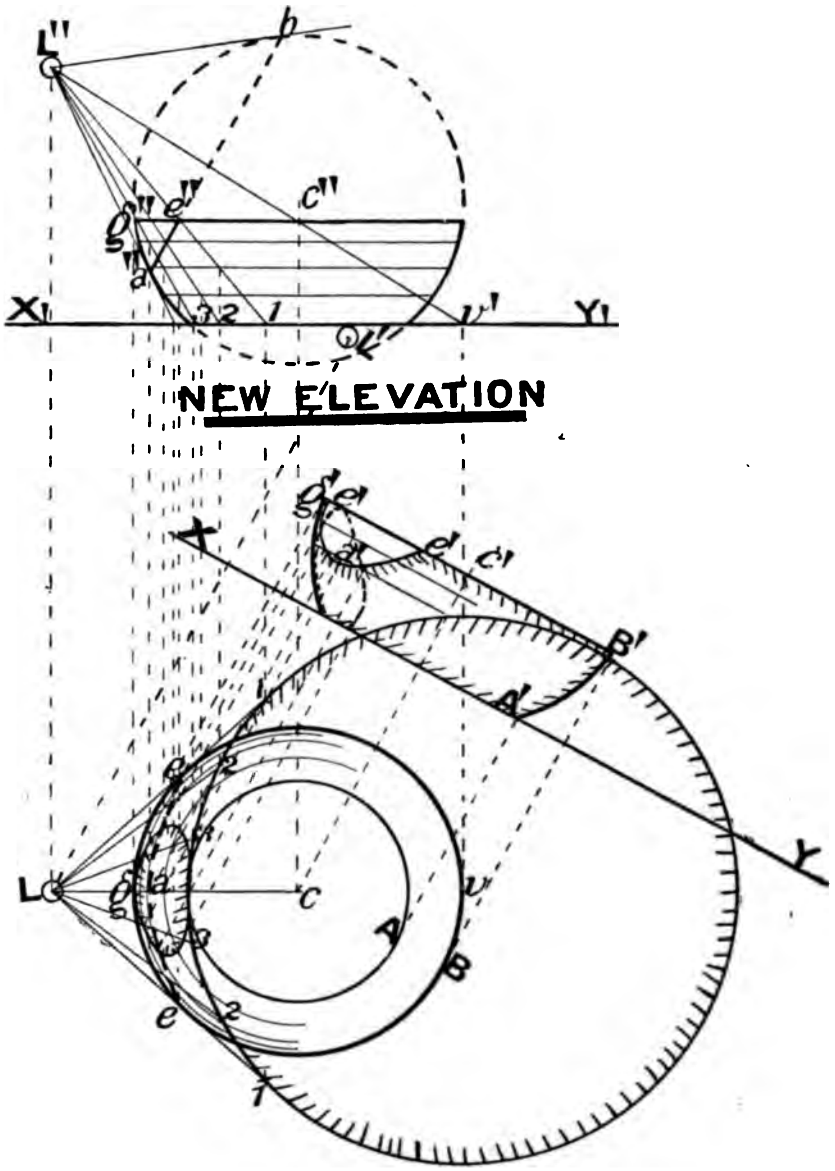
No. 61, 1892. **The plan and elevation of a bowl of a spherical form standing on the horizontal plane are given. Assuming the point L as the source of light, determinethe shadow of the bowl on the horizontal plane and the portions of the bowl in shade, both in plan and elevation.**

Let the given figure be represented in plan by the two circles A and B with centre *c*, and L the source of light, and the elevation on XY at $A'B_1'L'$.

The working of all problems of this kind will be greatly facilitated if a new elevation of the whole is made on X_1Y_1, taken parallel to the vertical plane passing through L, the source of light, and *c*, the centre of the bowl, as shown, and drawing the complete sphere of which the bowl is a part, and imagining a cone of rays of light enveloping this sphere (vertex at L). Then the line of demarcation which casts the shadow on the horizontal plane will be the circle of contact of cone and sphere at $a''e''b$; but as the bowl is a part of sphere only, clearly a part only of the circle of contact will cast shadow, *i.e.* from a'' to e'' in elevation and the curve *eae* in plan, and the remaining portion of circular top of bowl. The shadows cast by horizontal circles on a horizontal plane, either by parallel rays or rays emerging from a point as in this question, will also be circles; for if we consider the rays enveloping the circle as a cylinder or a cone, a horizontal section through the rays in question will be a circle, hence all sections parallel to it will be circles. The shadow of the part of the circle of contact a'' to e'' on the HP, is easily found by taking a series of horizontal sections through the bowl, giving circles in plan, thus: The circle of contact cuts the top of the bowl in two points, represented in elevation by e'' and plan at *ee*.* The horizontal trace of the rays of light through $e''ec$, as at 1 1, will be two points in the plan of shadow; similarly for 2 and 3. The curve 1 2 3 and the segment of circle centre *v* will be the complete plan of shadow on the horizontal plane.

---

* See p. 292.

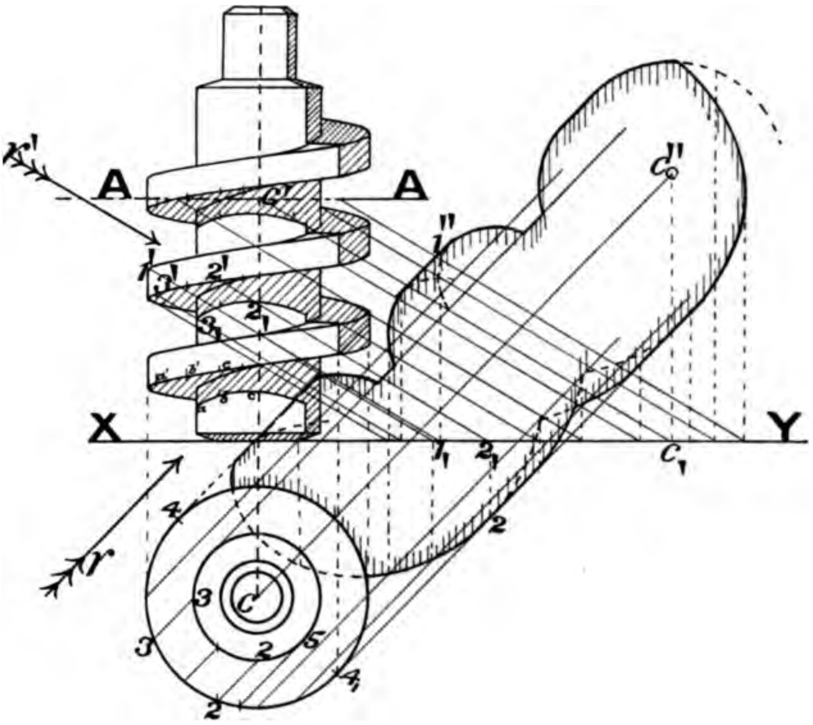

L''
b
c''
e''
g''
a''
X1
3
2
1
v'
Y1
L'
NEW ELEVATION
g'
e'
d
c'
a'
X
B'
A'
Y
L
g
a
c
v
A
B
e
2
1

Next let us consider the parts of the bowl in shade, on its exterior surface. Evidently, from the previous working, the part of sphere on the light side of the line of demarcation, $a''e''b''$, will be in full light, while the opposite side of $a''e''b''$ will be in shade. Then determine the plan of the surface between $a''e''$, as shown by the dotted line aee, which is readily found, as it is part plan of the circle $a''e''b$, which of course is an ellipse, by alternating the plans of the crossing-points of the circle with the horizontal, sections already taken through the bowl. This curve, projected to the given elevation (on xy), at $a'e'e'$, will be the part of the exterior surface exposed to light, while the remainder will be in shade.

Lastly, let us consider the interior surface of the bowl, which will differ from the exterior in so far that the largest portion will be exposed to light, while a very small portion will be in shade, and will be represented in plan by the curve shown, which may be found by taking a series of vertical sections of the bowl by planes passing through the illuminating point L, these sections rabated to the ground, when the ray touching the rim will be found to cut the said sections in the required shadow curve. The plans of these points being found, the elevation is readily determined, the height of every point in the shadow being found by the rabatted sections. The working is not shown here, but the careful student will readily follow it.

No. 61, 1893. **Determine the shadows on the screw, the rays of light being in the direction shown by the arrows, in plan and elevation. Determine, also, the shadows thrown on the horizontal plane by the portion of the screw below the line AA.**

The shadows on the screw will be found by taking a series of vertical planes parallel to the rays of light, cutting the screw thread in sections, and if rays drawn in these planes parallel to the given ray from points in the thread intersect the sections, these points will be points in the shadow.



Thus take planes tangential to the thread in plan at 4 4; then the part beyond these points will be in shade. Project these points to the elevation, giving the contact line of the planes with the thread, which of course will be the line of demarcation.

Take, again, the plane tangential to the inner cylinder at point 5: the contact line of this plane with that cylinder will also be the line of demarcation.

Now consider the planes cutting through points 3 3 and 2 2, and determine their intersections with the inner cylinder; then draw the rays, such as 3'3' and 2'2', from each thread, intersecting the section line on the inner cylinder on 3_1 and 2_1, etc.: these will be points in the shadow curve.

Similarly, take a sufficient number of points, as *a*, *b*, *c*, etc., to determine the true shape of the shadow thrown on the inner cylinder, which is shown scored in elevation.

To determine the shadow on a horizontal plane, take a series of lines parallel to the rays of light from every possible point on both edges of the thread, *i.e.* the upper and lower respectively. Their horizontal traces will be points in the shadow thrown on the H.P.; then the curve drawn through the exterior points will be the shadow required.

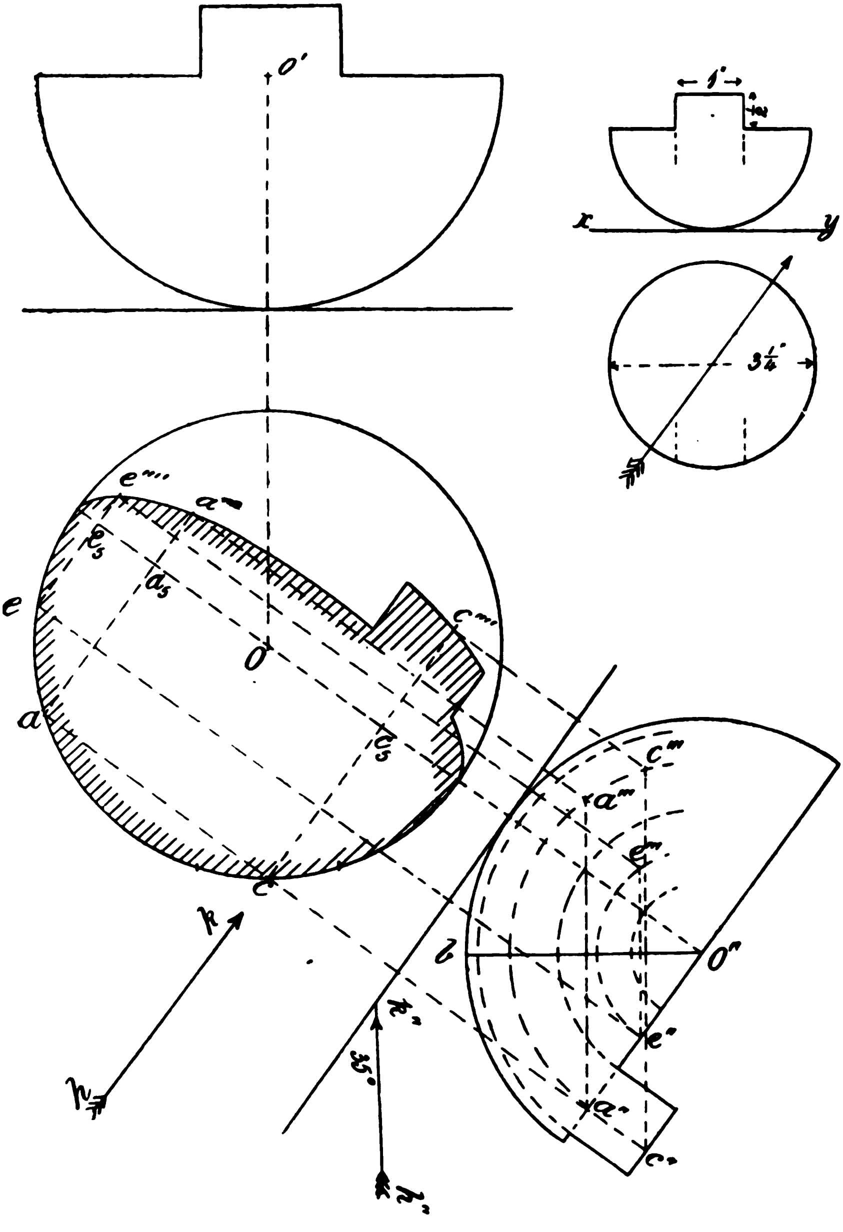
However, the student must select points which are most convenient. For instance, all points in the line of demarcation are points which will cast a shadow on the horizontal plane, when points on the farthest side will not cast any shadow, and few points on the threads between the tangent planes to the outside and inside cylinder may be selected.

The shadows of the top of the cylinders cut by the horizontal plane AA would of course, if they were complete circles, be circles at s', partly shown by the dotted arc.

No. 62, 1890. **A hemispherical vessel has a projection on its rim as shown. Draw the vessel full size in accordance with the given dimensions, and determine in plan the outline of the shadow cast upon its concave surface. The rays are parallel, and inclined to the ground at 35°. (Small sketch.)**

The plan of a ray is shown at hk, and its elevation at $h''k''$ on a ground line parallel to the plan of the ray. The shadow of the point a will lie in the line aa''''. Cut the vessel with a vertical plane passing through aa''''. The elevation of the section will be circle $a''a'''$. Through a'' draw a line parallel to the elevation of the ray, cutting the circle in a'''. Then a'''', the plan of a''', is the shadow of a.

Proceeding in this way, the shaded outline is obtained.

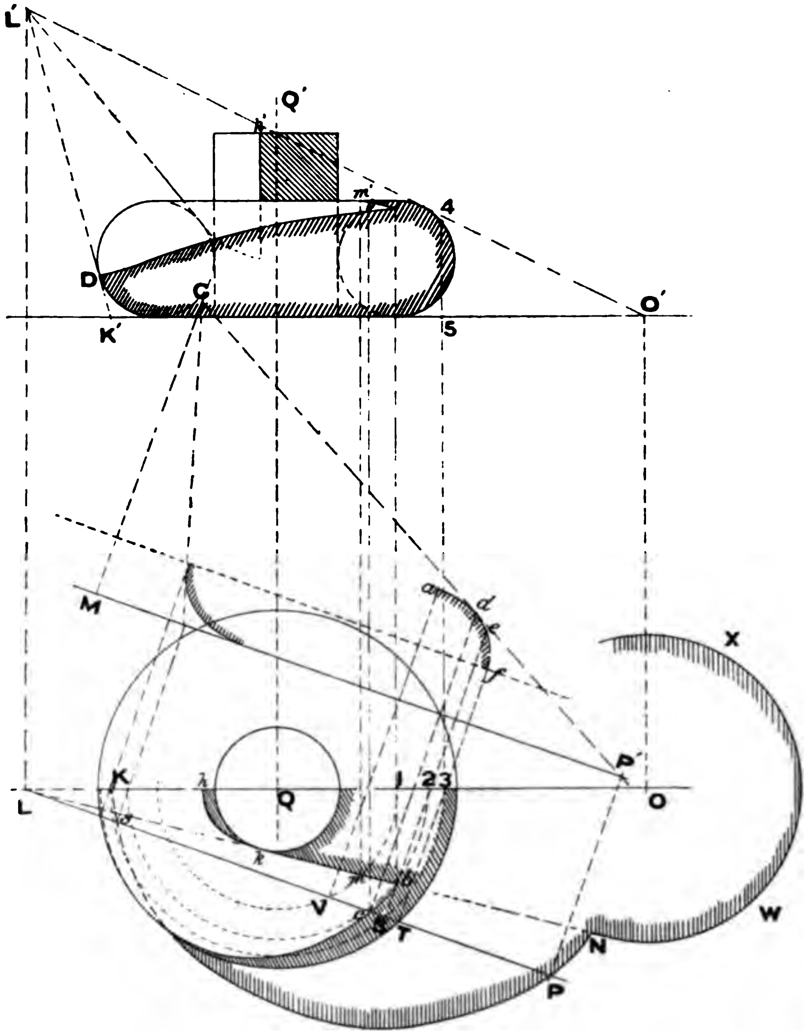

o'
1"
x
y
3 1/4"
e'''
a'''
e₅
a₅
e
c'''
O
a
c₅
c
p
p
p''
35°
p₂
c'''
a'''
e'''
b
O''
e''
a''
c''

No. 61, 1894. **The plan of an anchor-ring resting on the horizontal plane is given. A vertical right cylinder (height 1·6 in.) standing on the horizontal plane closely fits into the ring. Determine the shadow cast on the horizontal plane and the unilluminated portion of the anchor-ring, taking the source of light at the point L, whose height is 2·7 in.**

Join L to Q, the plan of the axis of the ring, and draw a ground-line K′O′ parallel to it. Describe on K′O′ as ground-line the anchor-ring and cylinder in elevation. Through L draw a tangent L*k*N to the plan of the cylinder, the contact point being *k*. All of the surface of the cylinder to the right of *k* will be in shadow; hence through *k*′ in elevation draw a vertical line, and the shaded area to the right shows the surface in shadow.

Also all the surface of the ring in plan, lying between the tangent LN and its companion tangent (not shown), and situated to the right of *k*, must be in shadow. Only half the shadow is shown in plan, as the other half is exactly similar and symmetrical.

Take a number of vertical sections of the ring, all of which pass through L, and one of which is LTP in plan. Take a new ground-line MP′ parallel to LP, and on it construct an elevation of the section, or that part of it which is required, such as the portions of the outline shown shaded. To obtain the outline, draw a number of circles with Q as centre; these are shown dotted. Two of these circles intersect the plan of the plane of section in V and S. The height of the point V above the ground is the same as the height of the point 1, and the height of S is the same as that of 3. The height of 3 is obtained by projecting upwards to 4. Then 5 4 is the height of S. Put up this height above MP′, and the point *e* is obtained. Find about four points in this way, and draw a tangent to the curve through these points, the tangent to pass through the new elevation C of the source of light. The point of contact is *d*. Project downwards perpendicular to MP′ through *d*, cutting LP in *c*; then *c* is a point in the outline of the shadow. Points



on the left-hand side of the annulus will be found in a similar manner. The horizontal trace of the line LC is at P, which is, consequently, a point on the outline of the shadow of the annulus on the ground.

The shadow of the top of the cylinder must be a circle whose centre is at O, and whose radius may be obtained by finding the horizontal trace of the ray through the point in the top which lies to the extreme right or left. The distance of this trace from O is the radius.

The elevation of the shadow is not asked for in the question, but is given in the diagram. There is also a small bit of shadow at *h*, which is obtained in the same manner.

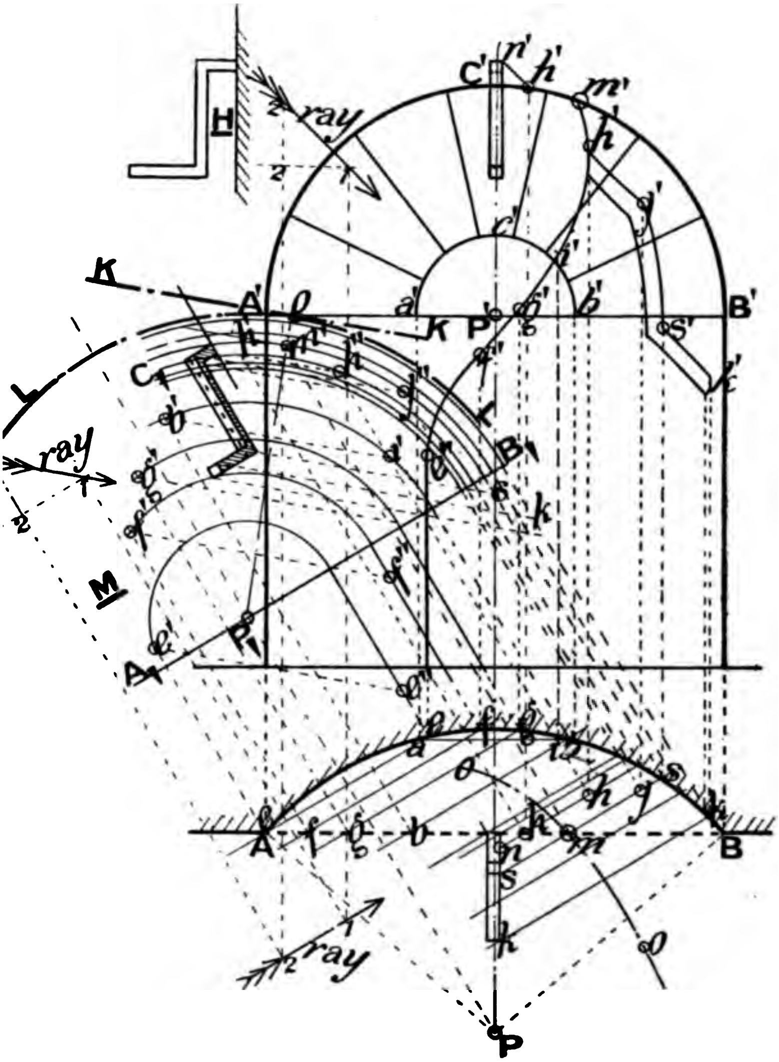
No. 58, 1895. **The diagram represents elevation and plan of a niche with a bar for a hanging lamp in front (see also side view). The curvature of the upper portion of the niche is such that every vertical section, parallel to AB in plan, is a semicircle in elevation; thus AB and *ab* in plan have for elevation A′C′B′ and *a′c′b′*. Draw the limiting outlines of the shadows thrown on the elevation of the niche (by its own contour) and the shadow thrown upon it by the rod, the parallel rays of light falling in the direction given by the arrows. Show how the point is determined where the curve of shadow cuts the outline A′C′B′.**

The plan, elevation, and section (Fig. $\overline{\mathrm{H}}$) with the projections of the ray of light, are given, as shown in thick lines.

The easiest method of solution in this and similar cases* is to make a new elevation on a ground-line, AB, parallel to the ray of light, as shown at Fig. $\overline{\mathrm{M}}$. Every point in the outline of the shadow is first determined in this view, and then its height and position transferred to the original elevation. It resolves itself into taking a series of vertical sections of the niche by planes parallel to the plan of the ray of light. Then in the new elevation the intersection of a ray (through the uppermost point in that section) with the outline of that section gives a point on the outline of the shadow cast on the niche, and so on for a number of sections: thus, take the section through *bb*; its elevation above the springing line A_1B_1 is a circle whose centre is P_1, and below A_1B_1 a straight line. Project *b* to *b′* on this section, *b* being a point on the contour A′B′C′. Draw *b′i′* parallel to the ray of light, cutting the section above A_1B_1 in *i′*. Project *i′* to *i* in plan (to the horizontal trace of the plane of section *bb*). Now project up to the original

---

* It will be seen from the above description (in the question) that the curvature of the niche is evidently spherical, for the outline of the horizontal *section* is circular, with P as centre.



elevation to i' so much above $A'B'$ as it is above A_1B_1 in Fig. M. Repeat the process till a sufficient number of points is found.

N.B.—The point where the curve of shadow cuts the outline $A'C'B'$ is readily found thus: (Fig. $\overline{\text{M}}$). The outlines of the sphere of which the niche is a part is drawn in elevation at LL, centre P_1, and imagine it to be shrouded by rays of light parallel to the given ray (the upper generator

KK only is shown). Then the circle of contact of this cylinder and sphere would evidently be the line P_1O in elevation and the ellipse omo in plan. The point where this line of demarcation cuts the upper contour of the niche mm' will be the point required.

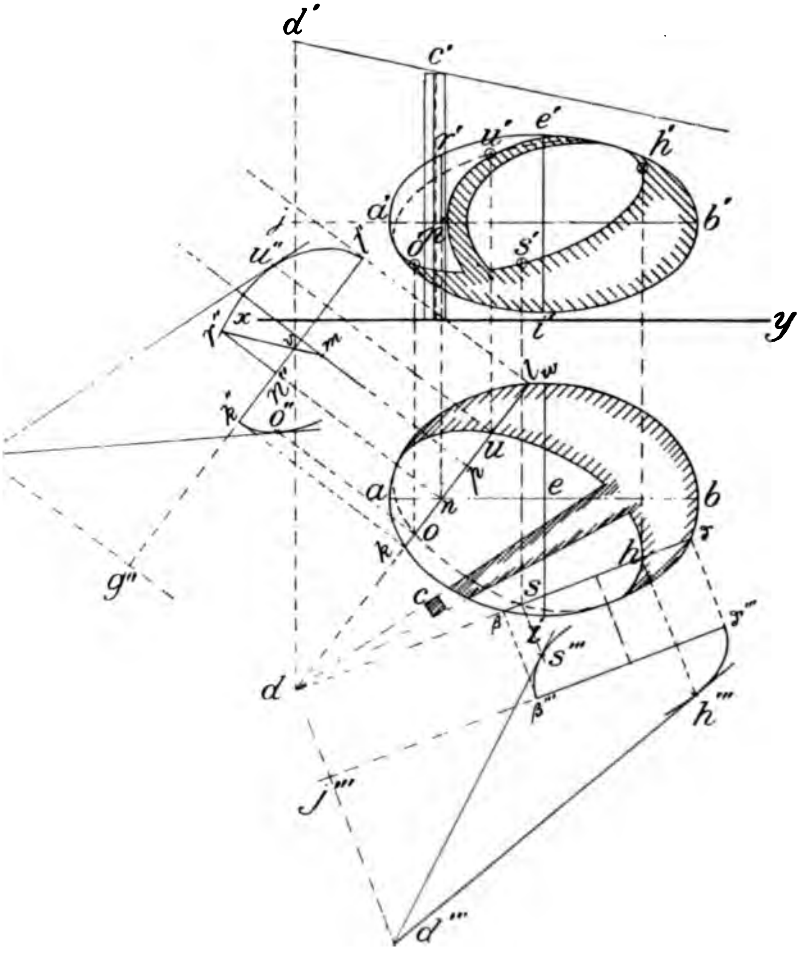
Similar reasoning will determine the shadow cast by the bar, but great care is required, and a large number of sections must be taken, of which a few are here shown; but the energetic and careful student will easily follow the drawing.

The shadow cast by the niche itself is the space included between the left of the curve $e''f'g'i'h'm'$ and the outline. The shadow cast by the bar is shown between the curved lines h', j, s', k'. It will be easier determined if the complete shadow of the horizontal bar is found first, then the vertical bar, and lastly the top piece, its shadow cutting the contour of shadow of $A'B'C'$ in $h'h$.

No. 58, 1896. **The axes of an ellipsoid are ab, wz, and $e'i'$, respectively equal to $3\frac{1}{2}$ in., $2\frac{5}{8}$ in. and 2in. Draw the ellipsoid and find the projections of the limits of light and shade upon it, and of the shadow cast on the solid by the square vertical rod cc'. The rays of light radiate from a luminous point dd'.**

First draw the outline of the ellipsoid in plan and elevation, and this may be most easily done by means of a slip of paper on which is marked the semi-major and minor axis from one end of it, then using the two points just found as the centres of the sliders in an elliptic trammels, mark off points on the ellipse which is traced out by the end of the paper slip.

Having traced out the ellipsoid, obtain the outline of the shadow on the ellipsoid. Cut the ellipsoid by a vertical plane which contains the luminous point. The true form of the section is shown in part at $l''u''r''k''o''$. This is obtained thus:—It is an ellipse whose major axis is $k''l'' = kl$. Only two points on this ellipse are already known, one of which is n. Its height above the major axis is $n'r'$. Set off $n''r'' = n'r'$, and, with r'' as centre and kp as radius describe an arc cutting pm in m. Make the



edge of the slip of paper coincide with $r''m$, the end falling on r'', and mark on it v and m. Then describe the ellipse in the usual manner.

Through d'' draw tangents to the ellipse, the tangent-points being u'' and o''. These are two points on the outline of the shadow. In plan they are at u and o, and in elevation at u' and o'. In the same way take a number of other sections, such as $\beta\delta$, and follow the same construction with each, finally obtaining the ellipses $uhso$ and $u'h's'o'$.

The shadow of the rod on the ellipsoid is obtained in the same way.

CHAPTER VIII.

PERSPECTIVE.

PERSPECTIVE differs from orthographic projection in several important points, which the student should endeavour to realize.

In "orthographic projection" it is always necessary to have both plan and elevation drawn, each of which gives the true or real forms of the object represented. All the projectors are drawn parallel to each other, and at right angles with the "plane of projection;" consequently the distance the object is placed from the observer's "eye" does not in any way alter the form of the projection of the object, while in perspective projection the pictorial representation of the object is obtained as it would actually appear to an observer situated at a fixed point in which all the projectors converge.

It is clear, therefore, that the farther the object is placed from this "point," the smaller its delineation on the plane of projection will be.

To illustrate this, let the student imagine himself viewing an object through a pane of glass, the outline of which is distinctly visible, and suppose threads were fixed to every corner of the object and brought to the "eye," thus forming a pyramid or a cone. Now, if the points where the threads intersect the pane be joined, the view thus obtained is termed a "perspective" of the object, which signifies "looking through."

Now suppose the pane to be moved farther away from the "eye," the picture would evidently be larger; and if moved towards the "eye," it would be smaller; from this it is clear that the perspective drawing of an object is the section made by the "transparent plane" with the pyramid above mentioned.

It is necessary, in order to obtain correct perspective drawing, that the "eye," the "plane," and the "object" should remain stationary, and occupy fixed positions with respect to one another. The "eye" must be considered as a point, and one eye only is to be used.

The following definitions have reference to Fig. 1 :—

Picture plane. The *picture plane* is supposed to be a transparent plane, through which the object is viewed, and is generally assumed to be placed in a vertical position between* the spectator and the object, provided it cuts all the projectors proceeding from the object and converging in the "eye."

Visual rays. The lines or projectors drawn from the object and converging in the eye may be considered as rays of light emanating from the object, and are called "visual rays" (such as S*a*, S*b*, S*c*, S*d*).

Ground-line. The intersecting line of the picture plane with the ground plane is called the "ground-line" (such as XY).

Centre of vision. The orthographic projection of the "eye" on the picture plane is termed the "centre of vision" (such as CV).

Principal visual ray. The line drawn through the centre of vision, and perpendicular to the picture plane is termed "the line of direction," "central visual ray," or "principal visual ray" (such as CV, S).

Station point. The point exactly opposite the centre of vision, in front of the picture plane, from which the observer views the object to be drawn, is termed "station point" or the "eye" (such as at S).

Visual angle. The angle subtended by the two outer rays is termed the "visual angle." This angle is generally limited to 60°, because the eye is unable to obtain a distinct view of a greater range of vision.

Horizon. The line which is the extreme limit of the ground plane, or where the earth and sky appear to meet, drawn in the picture through the centre of vision, is called the "horizon," or horizontal line; it is the line in which all horizontal lines vanish, except those parallel to the picture plane, which have no vanishing point.

Vanishing point. The point to which parallel lines appear to converge is called a "vanishing point;" it is found by drawing a line through the "eye," parallel to the original line until it intersects the picture plane (as VP, VP_1).

---

* In some instances the object is placed in front of the picture plane (see Quest. 53, 1889). In such cases the "projection" of the object or any portion of the object which shall be in front of the picture plane will be greater than that portion itself, but the construction remains precisely the same, as in the following cases.

But it may be mentioned that, under these conditions, the projection, although true in form, will not be "perspective" proper, "*which signifies looking through a transparent plane,*" but a section made by any plane (need not be transparent) of the rays; or, in other words, it will be the complete shadow of the object cast on the plane by rays of light emanating from a luminous point, in this case the eye.

Prob. 1. **A rectangular slab, 8 feet long, 3 feet wide, and 1 foot 6 in. thick, is shown by its plan, *abcd*. Draw it in perspective under the given conditions. XY is the ground-line; S the station point 12 feet from the picture plane, and 6 feet above the ground plane (draw to a scale of $\frac{1}{24}$ full size).**

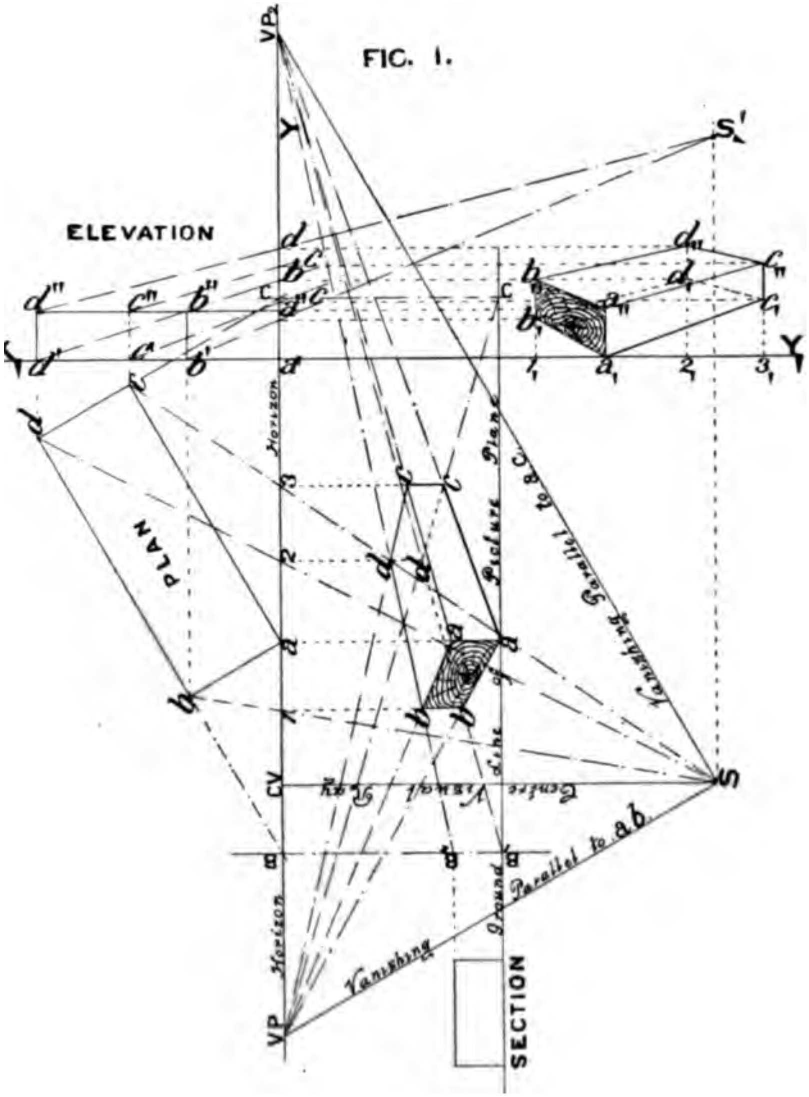
(*a*) By simple orthographic projection.

(*b*) By the use of vanishing points.

(*a*) Commence by drawing the plans of the visual rays to every corner of the object, as S*a*, S*b*, S*c*, S*d*, cutting the ground-line XY in points 1, *a*, 2, 3. Then the distance 1 . . 3 measures the width of the perspective projection of the object on the picture plane. It is clear, therefore, that the projection of the vertical edges of the solid will be contained by vertical lines, at distances away equal to 1, *a*, 2, 3.

Now, it is only necessary to obtain the heights at which the rays appear to cut the plane of the picture. To do this, draw X_1Y_1 at right angles to XY and parallel to the principal visual ray SCV, in any convenient position, and draw the elevation of the slab, the picture plane, the point of sight S′, and the visual rays, in respect to this new view. The slab is shown at $a'a''$, $b'b''$, $c'c''$, $d'd''$; the picture plane is shown by the line aYVP$_2$, which is the continuation of XY; the visual rays are shown by $S'a'$, $S'a''$, $S'b'$, $S'b''$, . . . $S'd''$, cutting the side view of the picture plane in points *cc* . . . *dd*, etc. Then these points represent the heights of each corner above the ground-line respectively.

To construct the perspective drawing, take a slip of paper and mark off the distances 1, *a*, 2, 3, and transfer to any convenient position on your paper, in this case on X_1Y_1 at 1_1, a_1, 2_1, 3_1. Now erect a vertical projector through each point to cut the horizontal projector from the corresponding point on the side view of the picture plane, *e.g.* vertical $2_1d_1d_{11}$, cuts the horizontal dd_{11} in d_{11}, $1_1b_1b_{11}$ cuts the horizontals $bb_{11}b_1$ in b_{11} and b_1 respectively, and so on for every point. These points being joined in regular order, the complete perspective projection is obtained.


FIC. 1.

The above process is applicable to all cases, but where the object is very complicated, it will be found very tedious to treat each point separately, as is necessarily the case here, and it will be found expedient to use the following method of vanishing points.

(*b*) *Second method. By vanishing points.*

For convenience, we may use the same diagram as before, ignoring, however, the construction lines of the last method.

Fig. 1. Let us consider the ground-line XY as "horizon" for the time being, and at 6 feet below draw a parallel line to represent the "ground-line" of the picture plane. The student may note that he is at perfect liberty to draw these at will, provided their direction be parallel to the principal visual ray, and their distance apart equal to the given height of the eye.

Through S, the station point, draw lines parallel to the side *ac* and the end *ab* respectively, cutting the horizon line in VP$_2$ and VP. These lines are called vanishing parallels, and the point VP is the vanishing point of all lines parallel to *ab*, and VP$_2$ is the vanishing point of all lines parallel to *ac*.

Let B*d*, *ac* be parallel lines lying on the ground plane, and let S be the station point, also on the ground plane, and XY to represent the ground-line of the picture plane.

Take any point *b* in line B*d*; join *b*S, cutting XY in point 1. Then 1 is the perspective of the point *b*.

Now suppose point *b* to move to the right along the line B*d* to *d*. Join *d*S, cutting XY in point 2. Then point 2 is the perspective of point *d*, and if supposed to have moved to an indefinite distance to the right, then the line Sb_1Sd_1 becomes parallel to B*d*, as shown by the vanishing parallel S VP$_2$.

By similar reasoning, it will be seen that VP$_2$ is the vanishing point of all lines parallel to bd_1ac, and VP$_1$ of all lines parallel to ab_1cd.

For convenience, draw the elevation or a section of the object in its proper position relative to the ground-line of picture plane, so that the various heights may be readily projected.

It will be seen that the vertical edge *a* of the solid is in the picture plane; draw, therefore, the perpendicular *aa*, cutting the ground-line in *a*. Through *a* on the ground-line draw *a*VP and $a\text{VP}_2$ through points 1 and 3 on the horizon; draw perpendicular to cut these lines in *b* and *c* respectively. Again, draw *c*VP and $b\text{VP}_2$, cutting each other in point *d*. Then *abcd* is the perspective projection of the bottom of the slab. Similarly proceed to obtain the upper face of the slab, thus: mark off *aa* equal to the thickness of the slab, and draw lines to the vanishing points as shown.

A slight variation of the last method may with advantage be adopted, either separate or in conjunction with it.

Imagine the vertical plane containing each of the vertical faces produced till they intersect the picture plane; thus the plane of the face *ac* and *ab* intersect the picture plane in line *aa*. On this line of intersection all the heights are to be placed, and then lines drawn to the vanishing points, as at *aa*. Again, the plane of the face *db* intersect the picture plane in line B_1B_2B, and that of *de* in line CC. Now through B_1 and C, the points of intersection on the ground-line, draw $B_1\text{VP}_2$ and CVP, cutting each other in point *d* as before; similarly, measure off the height B_1B_2 equal to the thickness of the slab, and draw the line $B_2\text{VP}_2$; similarly deal with line from C. The drawing thus obtained will be exactly the same as before.

It will be seen that the drawing thus found is the solid enclosed by the four vertical planes containing the faces, two of which vanish in VP, and the other two in VP_2, their lines of intersection forming the edges of the solid.

The use of the vanishing planes will be found useful where a number of points are contained by the plane, or where a curved surface has to be dealt with (see Quest. 63, 1891, and 59, 1895).

The junction of the king post with the tie beam and a short length of the strut, making an angle of 45° with the king post, is shown by the plan and by the elevation A. Draw the perspective under the given conditions. HH is the horizontal line, S the station point, and GG the ground-line.

(*a*) *By the first method.*

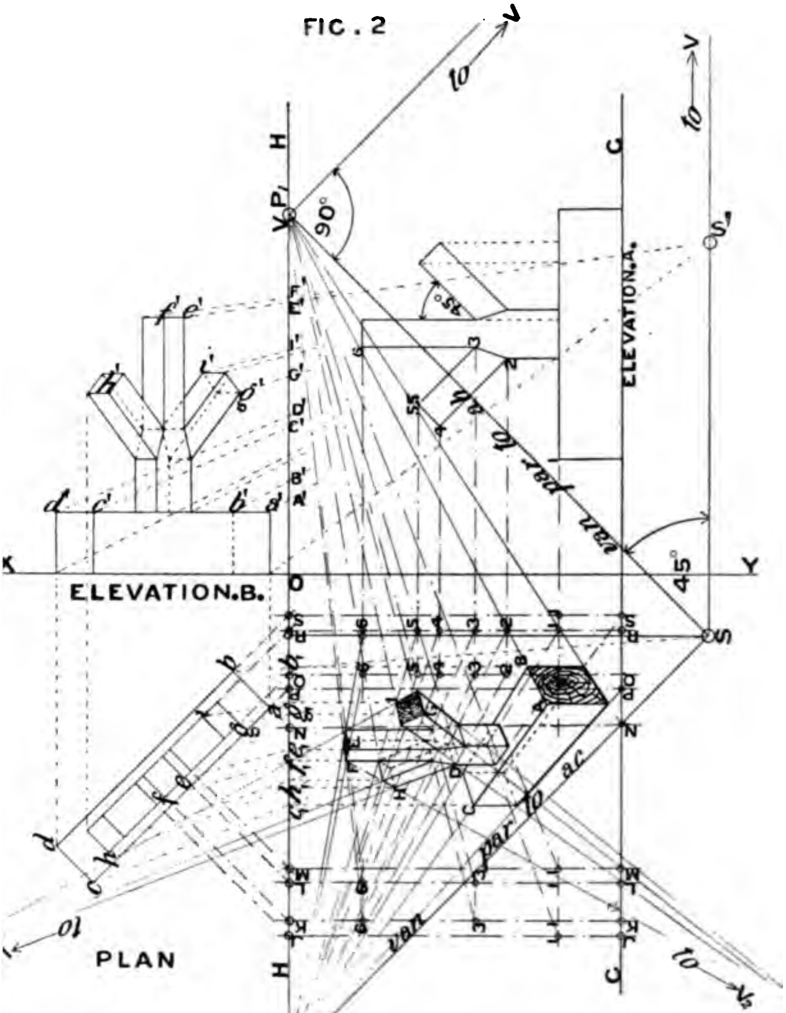
Commence by drawing XY at right angles with HH and make a new elevation B of the joint as shown; determine S′, and draw the projectors to every corner of th solid converging to the eye, both in plan and elevatio Transfer the heights of each corner of the solid to its prope projector perpendicular to HH, as shown, in precisely th same way as in last problem. Thus the distance of corne A in the perspective from the ground-line GG equals OA in the side elevation above XY. Similarly, the distanc of corners B, C, D, I, G, E, F above the ground-line equal OB′, OC, OD′, OI′, OG, OE, OF in the side elevation abov XY, and so on for every corner of the solid. These point being carefully joined in regular order, the complete perspective projection will be obtained.

(*b*) *By the second method.*

Commence by drawing the vanishing parallels to *ab*, *a* through S, meeting the horizon in VP and VP_1. Then V is the vanishing point of all lines which are parallel to *a* and VP_1 is the vanishing point of all lines parallel to *ab*.

The plane of the face *ac* intersects the picture plane i line PP, and the plane of the face *bd* intersects it in lin SS, and the plane of the end *ab* intersects it in line N Now through P and S, where the lines of intersection c the planes cut the ground-line GG, draw lines to th vanishing point VP on the left, and from points 1_1, 1 on line PP, SS (the height of the bottom block as seen projecte over from the elevation A), draw lines to the same vanishing point as before. Similarly, the planes of the ends *a* and *cd* may be treated; their intersecting lines would, o course, form the complete bottom block: or draw the complete plan of the cone of rays, and draw the perpendicula projectors from their crossing-points with the picture plan to cut the vanishing lines in ABCD, *e.g.* the ray throug b cuts HH in b_1, and the ray through C cuts HH in c_1; the the projector through c_1b_1 cuts the vanishing lines 1_1VP i points C, A, etc.

FIG. 2



Next draw the plane containing the face *gefh*, and similarly the corresponding plane at the back, and proceed precisely as before, until the complete projection is obtained, always remembering that the heights in all instances are measured on the line of intersection of the plane containing those points with the picture plane.

It may be noted that the vanishing points V_2, V_3 of the struts may readily be found. Thus consider the plane of the face *gefh*, whose vanishing parallel is S,N,VP, and the struts making an angle of 45° with the king post.

If from the eye S lines be drawn at an angle of 45°, and contained by the plane of the vanishing parallel to *ac*, the point of intersection of these lines with the vanishing line through VP will be the vanishing points of the struts. For convenience in the present case, let us use the vanishing parallel on the right, SVP_1, as the base of the right-angle triangle thus formed; SVP_1, will of course be horizontal, and level with the eye. Imagine the plane of the face *gefh* rotated to the horizontal plane about this line. Now draw the vanishing parallel to the strut, making an angle of 45° with the level line SVP_1, as SS to V, cutting the perpendicular projector through VP_1 in V. Then the length VP_1V is the distance of the vanishing points from the horizon HH, both above and below VP on the vanishing line.

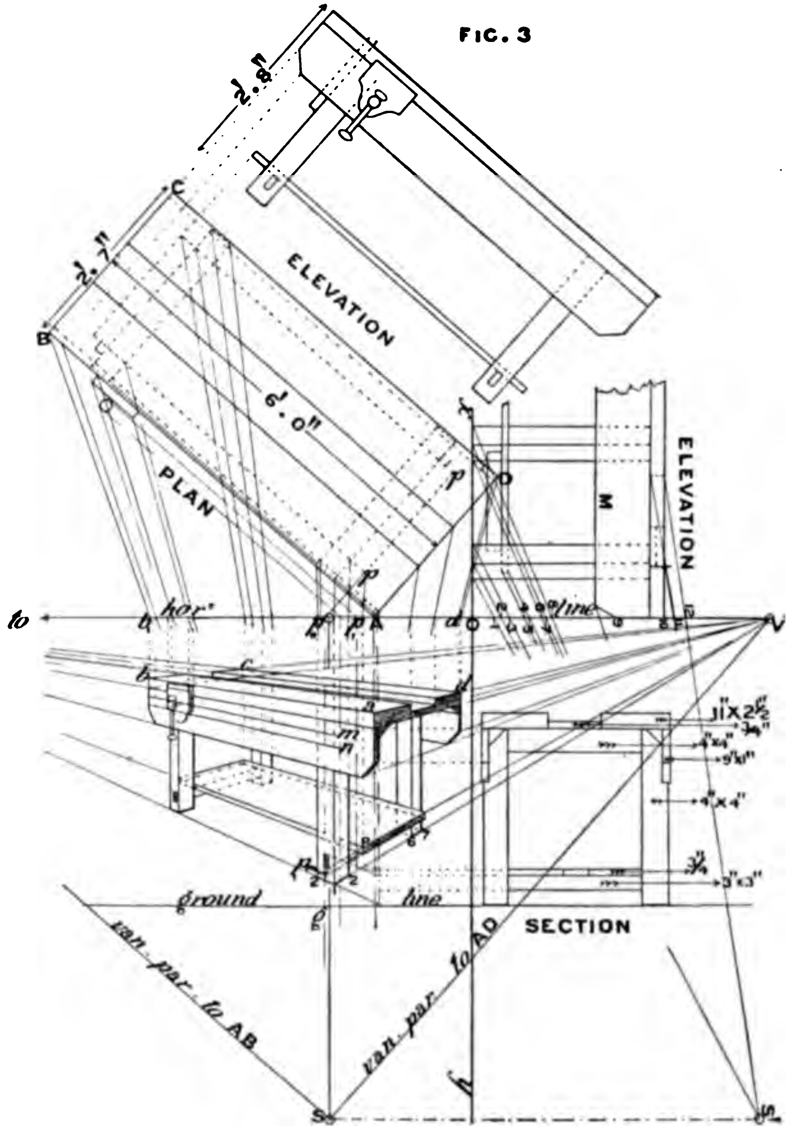
The student is recommended to work this example out to about four times the size of the drawing given, using both methods, and proving for himself that the results are precisely the same, when he will be able to adopt the best method for his own special case.

Fig. 8.

A carpenter's bench is given in plan and elevation. Draw it in perspective under the given conditions, using both methods.

Commence as before, by drawing the ground-line *xy* at right angles with the horizon line, and making an elevation of the bench relative to *xy*, as partly shown at M, and draw the cone of rays both in plan and elevation, and transfer the heights or the plans to any convenient position, as before explained.

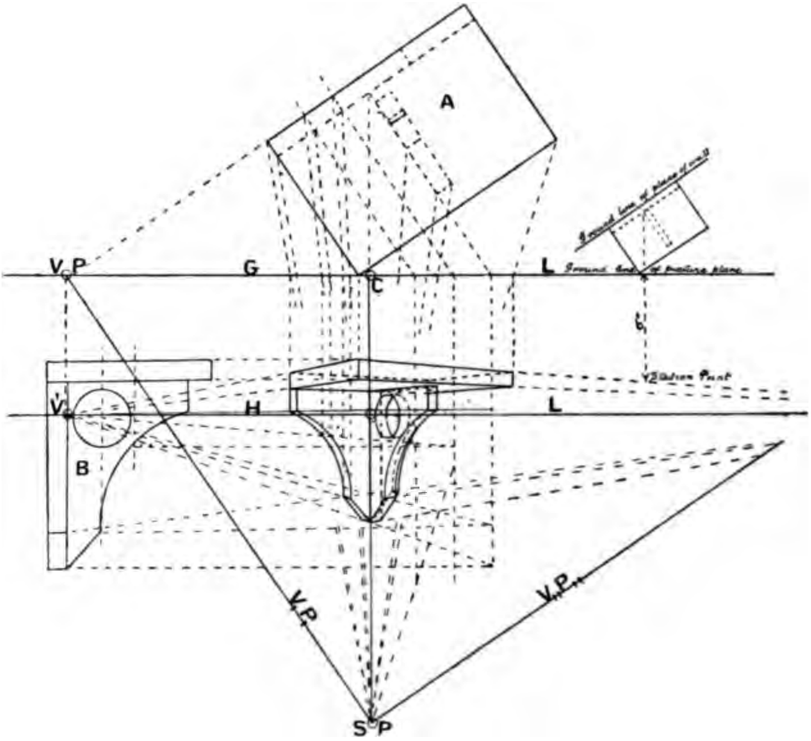
FIG. 3



The vanishing points are obtained by drawing the vanishing parallel to the side and end of the bench respectively, as at V, V_2.

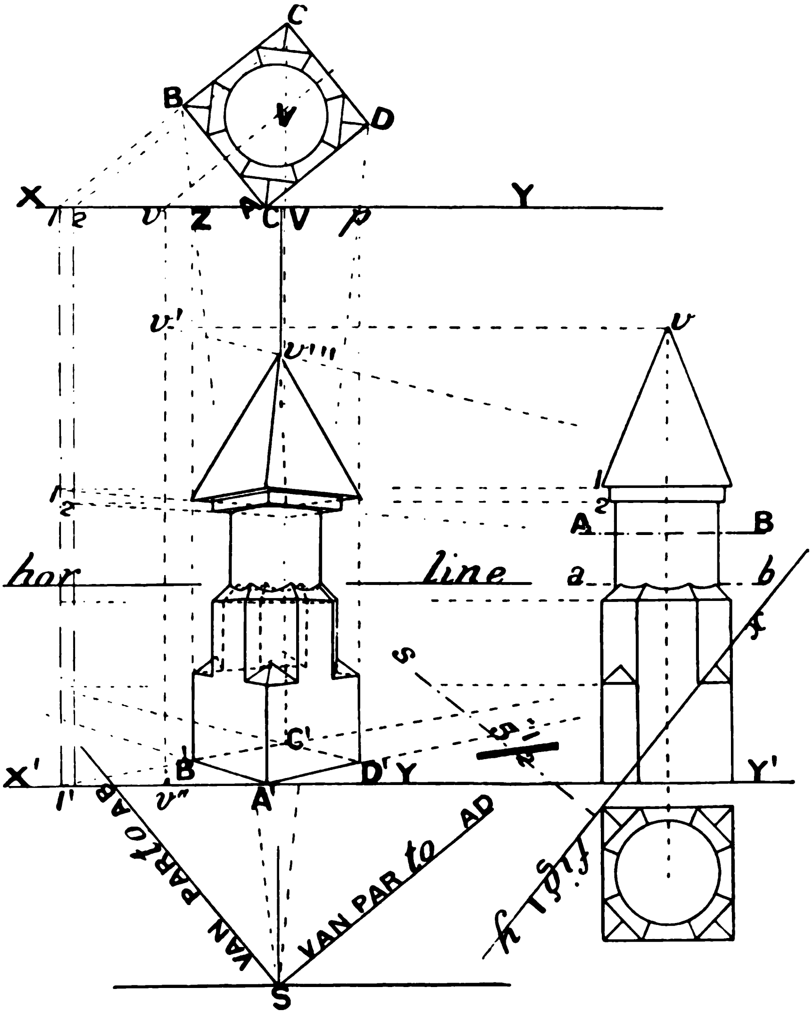
No. 53, 1887. **The sketch-plan shows the position of the given bracket with respect to the plane of the picture. The station point is 6 in. from the picture plane and $\frac{3}{4}$ in. bolow the level of the top of the bracket. Draw a perspective view of the bracket. N.B.—The sketch-plan is to be taken only as giving the position of the bracket.**

Draw the plan A in its proper position relative to the ground-line GL, and to the central visual ray C,SP. Make CS = 6 in., and draw V_1P_1 parallel to the left side of bracket, meeting GL in VP, and draw $V_{11}P_{11}$ parallel to the right side of bracket, also meeting the ground-line in a point VP, which is the vanishing point for all horizontal lines parallel to that side. Next draw the plan of the rays, cutting the GL in points which show the width of the object on the picture plane [in plan.] Now draw the horizontal line HL at any convenient position, and make the side elevation of the bracket as shown, the top $\frac{3}{4}$ in. above HL. The front corner is in the picture plane; its height is therefore taken direct. The edges vanish to the right and left respectively, and meeting the perpendicular from the point where the ray from the back corner to SP cuts GL, thus determining the width of the shelf from front to back. No detailed description is here necessary, as the lines are all shown—remembering, however, that all heights are determined on the trace of planes parallel to the sides of bracket on the picture plane, and vanishing to their respective points, cutting their projectors from GL.


A
Ground line of plane of wall
V.P.
G
C
L
Ground line of picture plane
6'
Station Point
V
H
L
B
V.P.
V.P.
S.P.

No. 53, 1888. **The elevation and plan at AB of a symmetrical tower are given. Make a perspective view of the tower, taking *xy* as the ground-line of the picture plane, and S as the plan of the station point, which is on a level with *ab*.**

The plan and elevation (Fig. 1), *xy*, the direction of the station point, and its distance away from *xy* (5½ in.) were given. The student may here notice that the elevation as given, is in a peculiar position regarding *xy*, and that it is easier in this case to transfer the plan to another position; therefore we may use the ground-line of the elevation X′Y′ for the ground-line of the picture plane, and through *ab* draw the horizontal line or horizon, which is the level of point S. Now draw XY parallel to X′Y′ in any convenient position, and draw the plan ABCD at its proper position relative to XY, and draw the central visual ray from CV perpendicular to XY to the station point S, and 5½ in. away from XY. Then from S draw all possible rays to every point on the plan of the tower, such as BZS cutting XY in point Z; also D*p*S, cutting XY in point *p*. The edge A is in the picture plane, therefore project point A to A′ in X′Y′, and through BC draw the trace of a vertical plane cutting XY in point 1. Project point 1 to 1′ on X′Y′, and from S draw the vanishing parallels to AD, AB in plan, meeting XY in *m* and *n*; these points projected back to the horizon, will be the vanishing points of all horizontal lines on the object. Therefore, from 1′ draw a line to the vanishing point on the right, cutting a vertical through Z in B′; also from A′ to the vanishing point, cutting vertical through *p* on XY in point D′; then A′B′ and A′D′ are the perspective of AB and AD. The corner A being in the plane of the picture, the heights may be projected direct to it from the elevation, and lines may be drawn therefrom to the vanishing point on the left and right for all other points contained by these vertical planes. Other points not in these planes may be similarly


C
B
D
A
X
Y
1
2
v
Z
C
V
p
v'
v'''
v
1
2
A
B
hor
line
a
b
x
s
5½
X'
1'
B'
v''
A'
C'
D'
Y
Y'
VAN PAR to AB
VAN PAR to AD
S
fig 4

found by taking vertical planes through them: thus, through V draw the trace of a plane Vv; then $v'v''$ is the trace of this plane in the picture plane. Take $v'v''$ the height of apex of the tower in elevation, and draw through v' to the vanishing point on the horizon, cutting the vertical from the point where the plan of ray SV cuts XY in point v'''; then v''' is the perspective of the apex of the tower. Following the same principle, every other point is obtained. The curves where the cylindrical part joins the octagonal part, planes are found in exactly the same way, by selecting a series of points and passing vertical planes through them.

Another method is here shown (Fig. 2), where the perspective view is obtained direct from the plan and elevation, *i.e.* by the ordinary methods of orthographic projection.* As before, draw the plans of the visual rays meeting XY in points 3, 4, 1, 2, etc. Next take X'Y' anywhere at right angles to X'Y', and make a new elevation of the tower as shown, and a new elevation of the station point S. Then draw the elevation of the rays from every point on the object to S', cutting the side view of the picture plane in points 1', 2', 3', 4', etc.; these are the heights at which the rays cut the picture plane. Now transfer the plans of the rays 3, 4, 1, 2, etc., to any convenient part of X'Y', such as 3", 1", 4", 2", etc., and erect verticals through these points to meet the horizontal projectors from the points 1', 2', 3', 4', etc., and on these points being joined, the perspective of the object is completed. This method is very easy where the points are few in number, and will be readily understood from the drawing; but where the object is very complicated, as in this case, the first method would prove the best.

---

* See previous problems.

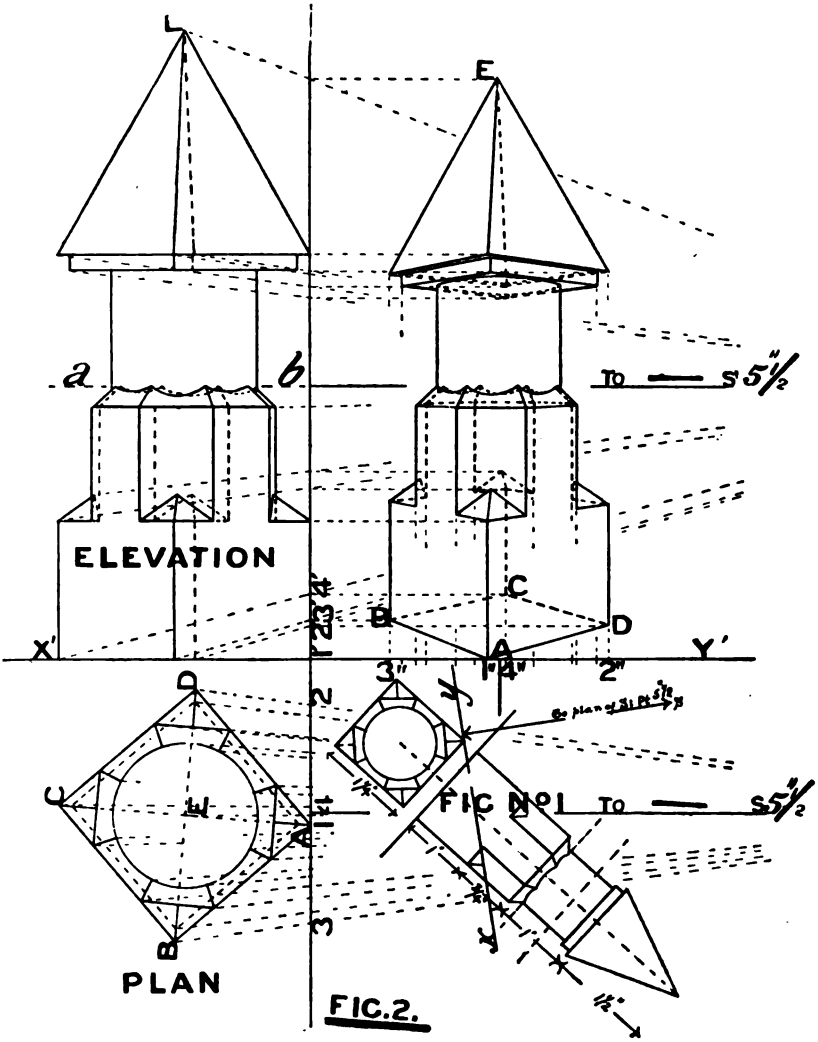

L
E
a
b
To S 5½"
ELEVATION
C
B
D
X'
A
Y'
3"
1"
4"
2"
1 2 3 4
2
1
3
D
C
E
A
B
PLAN
FIG No 1
To S 5½"
y
x
1¼"
1½"

FIG. 2.

No. 53, 1889. **Make a perspective view of the panel (given by its plan, Fig. 1, and by its elevation, Fig. 2) when standing vertical, with picture plane and station point as shown, the horizontal line being $3\frac{3}{4}$ in. above the bottom of panel.**

It will be noticed here that the panel is placed in front of the picture plane, therefore the view on the picture plane must be larger than the actual panel, and a portion of it will appear below the ground plane (provided the ground plane is considered transparent); however, applying the same method as before, the perspective is easily obtained. The plan of picture plane VV, plan of panel, and the station point S are given in their positions. Draw the plan of the rays from the station point to meet the picture plane in points such as 1, 2, 3 . . . 15, *m*; next draw from S the vanishing parallels of the face and end of panel to meet the plan of picture plane in V, V_2. Draw a horizontal line in any convenient position on your paper, such as VP, VP_2. Consider this line as the line in which all horizontal lines vanish, it being the level of the eye. Draw a line WW at a distance of $3\frac{3}{4}$ in. below it, to represent the plan of picture plane. Next transfer the distances V, CV, 1, 2, 3 . . . 15, *m*, V_2, on VV_2, where the plans of rays and the vanishing parallels meet the plan of picture plane, to the horizon VP, VP_2, as shown at VP, CV, 2′, 7′ . . . *m*′, VP_2. Draw the elevation of the panel on the ground-line WW, as shown, and draw horizontal lines from each point in this elevation to meet the intersecting line of the plane containing those points with the plane of picture, *eg.* The plane of the near face of panel meets the line VV in point *n*. Transfer the distance CV*n* to C′V′*n*′, on line $VPVP_2$, and draw the line $n'n_2$, which is the line of intersection of the two planes. Now draw the horizontals to meet it, thus, the top meets it in point *g*, and the bottom in n_2. Through *g* and n_2 draw the vanishing lines to VP_2, and produce forwards in front of the picture plane, to meet the perpendiculars from the corresponding points in VP, VP_2, *i.e.* the

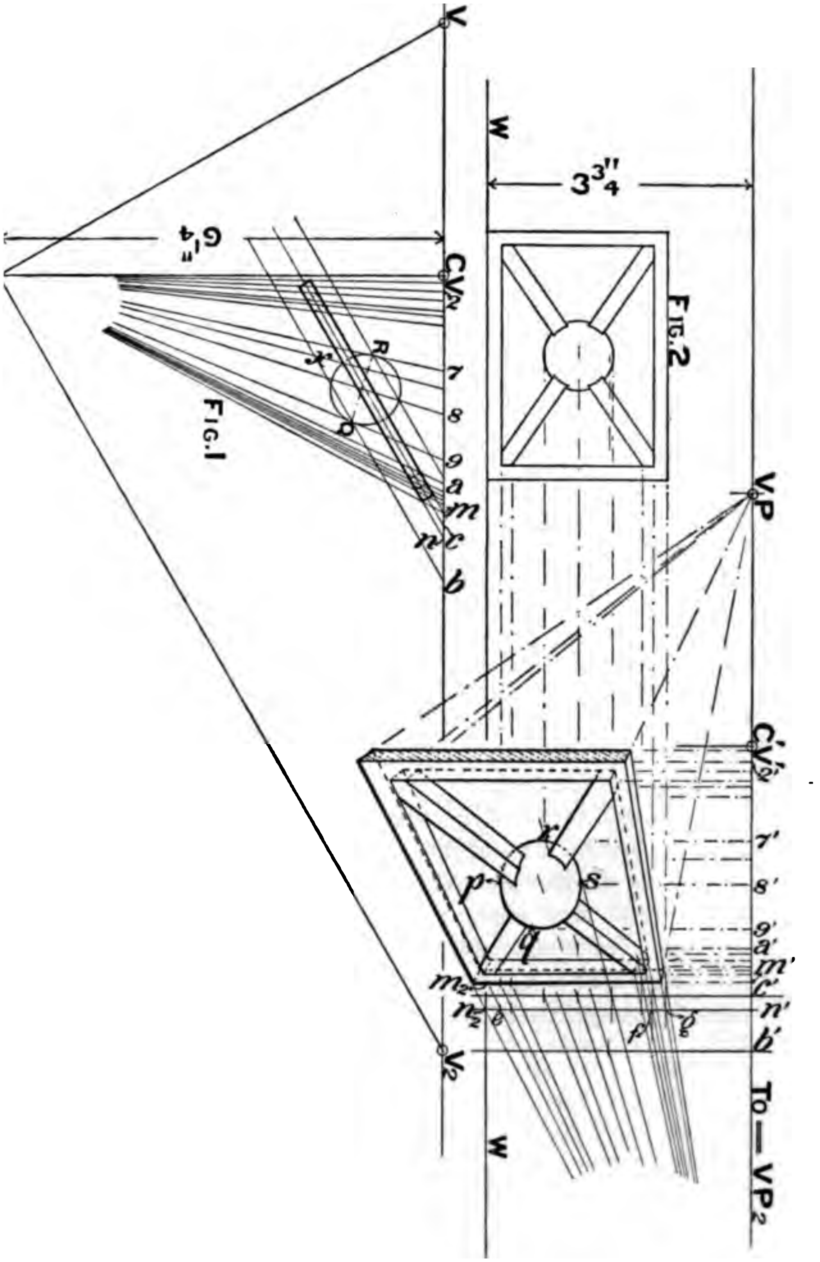

V
CV2
6 1/4"
W
3 3/4"
R
r
Q
7
8
9
a
m
n
c
b
VP
C'V'2
r'
8'
9'
a'
m'
c'
n'
b'
r
p
s
q
m2
n2
V2
W
To—VP2

FIG. 1

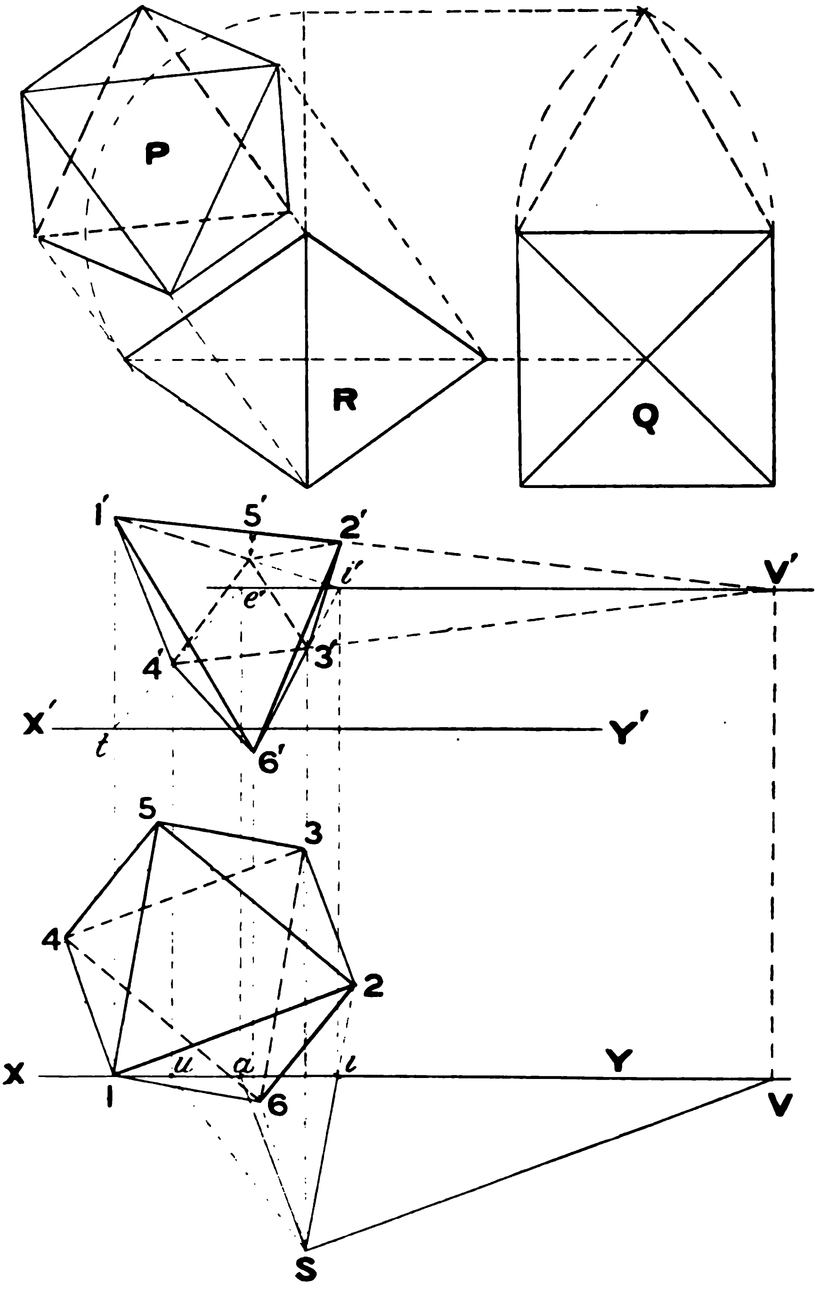
FIG. 2

width of face of the panel in perspective is $2'm'$, therefore through $2'm'$ draw the perpendiculars from these points, meeting the vanishing lines through g and n_2, thus completing the outline of the near face of panel. Proceed in exactly the same way to obtain all the other parts, remembering to measure all heights on the intersecting line of the plane passing through those points with the PP, and then drawing the lines to their proper vanishing points, and producing them till they meet the perpendiculars from the corresponding points. The heights of all points on the back surface of panel are measured on the line m_1, the height of R and Q are measured on lines a' and b' respectively, being the intersection of the planes containing them with the PP. The view of the sphere is found by obtaining the trace of the enveloping cone of rays in the picture plane. The line RQ represents the plan of the circle of contact of the rays with sphere. The points where the diagonal pieces meet the sphere may be obtained by drawing the circle on the face of the panel in perspective as shown. Further description is unnecessary, as all the lines are retained. The method of two views, or simple orthographic projection, would prove very convenient for problems of this kind (see Fig. 2, No. 53, 1888).

No. 63, 1890. **An octahedron of 2″ edge rests with one face on the ground plane. One of the angles of the upper face is in the plane of the picture, and one of the horizontal edges meeting at that angle makes 20° with the plane of the picture. Make any perspective view of the octahedron, using an horizon at a height equal to two-thirds of the distance apart of the parallel faces of the solid.**

Draw the plan of the octahedron as shown at Q, from which can then be obtained the elevation at R, and a further plan P, with one face resting on the ground.

The plan of the picture plane is taken at XY, and the ground-line at X′Y′ together with the horizon $e'i'V'$ at


P
R
Q
1′
5′
2′
i′
V′
e′
4′
3′
X′
Y′
t
6′
5
3
4
2
X
u
a
i
Y
1
6
V
S

a distance from X′Y′ equal to two-thirds of the distance apart of the parallel faces. The station point S is taken at random, because its position is not stated. The corner 1 of the octahedron is also taken at random in *XY*, and the line 1 2 is inclined to *XY* at 20°. The plan of the solid is then completed. The point 1 being in the picture plane, its perspective will be vertically above it at 1′, such that *t*1′ is its real height above the ground.

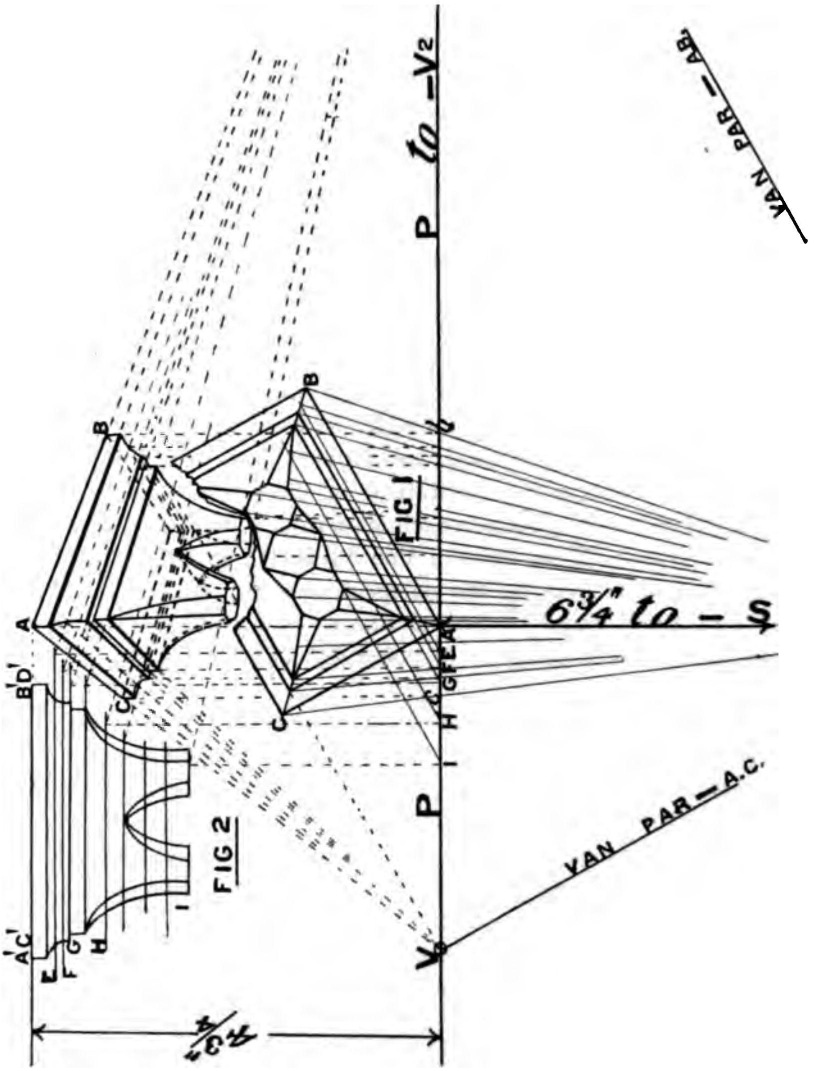
Next obtain the vanishing points. Select the greatest number of parallel lines. In this case 1 2 and 4 3 are selected, and draw through S the line SV (the vanishing parallel), parallel to 1 2, cutting XY in V. Project up to the horizon at V′. Through 1′ draw 1′V′, and join 2S, cutting *XY* in *i*. Project *i* upwards until it cuts 1′V′ in 2′. Similarly 4′3′ is the perspective of 4 3. The vanishing point of lines parallel to 1 5 is at *i*′, hence draw 1′*i*′. Join 5S, and from the point where it cuts *XY* project upwards until the projector cuts 1′*i*′ in 5′. The point 6′ is obtained in the same manner.

No. 63, 1891. **Make a perspective view of the capital, with picture plane and station point as shown, the horizontal line being $4\frac{3}{4}$ in. below top of capital.**

The plan of picture plane, labelled PP, the plan of capital ABCD, Fig. 1 (here only partly shown, so as not to obstruct the perspective view), the direction and distance to station point, and an elevation of the capital on *xy* parallel to CD, are given.

Let us commence by drawing the vanishing parallels AC and AB from station point S, cutting PP in points v and v_2: then v, v_2 will be the vanishing points of all horizontal lines on the object parallel to AB and AC; thus, AB will vanish to the right at point v_2, and all lines parallel to it will also vanish in point v_2; the line AC, and all lines parallel to it, will vanish to the left at V.

Next draw the plans of all possible visual rays from S



to the object; thus, the ray from corner B cuts PP in point *b*, and the ray from C cuts PP in point *c*. The length *cb* represents the total perspective width of the capital in plan. Similarly, draw a series of rays from every possible point on the outline of object. To obtain points on the curve, you must imagine a series of vertical planes cutting the capital parallel to AB, shown in plan, cutting PP in points E, F, G, H, I.

Now let us consider the line PP as the horizon, or "eye-level," and draw the elevation of the capital at any convenient position, with the top 4¾ in. above the horizon, as shown. Then the heights of all points are readily determined, if we consider every point as being contained in vertical planes parallel to AB; thus the plane containing AB has its intersection with the picture plane PP in a line drawn perpendicular to PP through A. The point A being in the picture plane, its height is fixed, and is on same level as top of capital, as shown; but the line AB vanishes to v_2, and is contained in the vertical plane just mentioned. Erect a perpendicular from b, the point where ray SB cuts PP, to cut the line AB in B; similarly for C: then AB, AC will be the perspective of top edges AB, AC of capital. From C draw a line to v_2, and from B draw a line to v, cutting it in point D, which completes the top of capital.

Again proceed to find other points in the same way, such as points contained in plane G. The intersection of this plane with picture plane will be the perpendicular line through G (in PP), cutting level line though G, Fig. 2, in point, which, on being joined to v_2, will give the top edge of the second fillet from top of capital. Perpendiculars from points in plan where rays cut PP will determine the perspective length of the lines.

Similarly, points in the curve are found in precisely the same way. Few only are here shown, for the sake of not confusing the drawing; but the student will readily follow out every point in it.

A second method is shown at Fig. 3. The operation employed is simply determining the points of intersection of a series of convergent lines with a plane.* Their plans

---

* See previous problems.

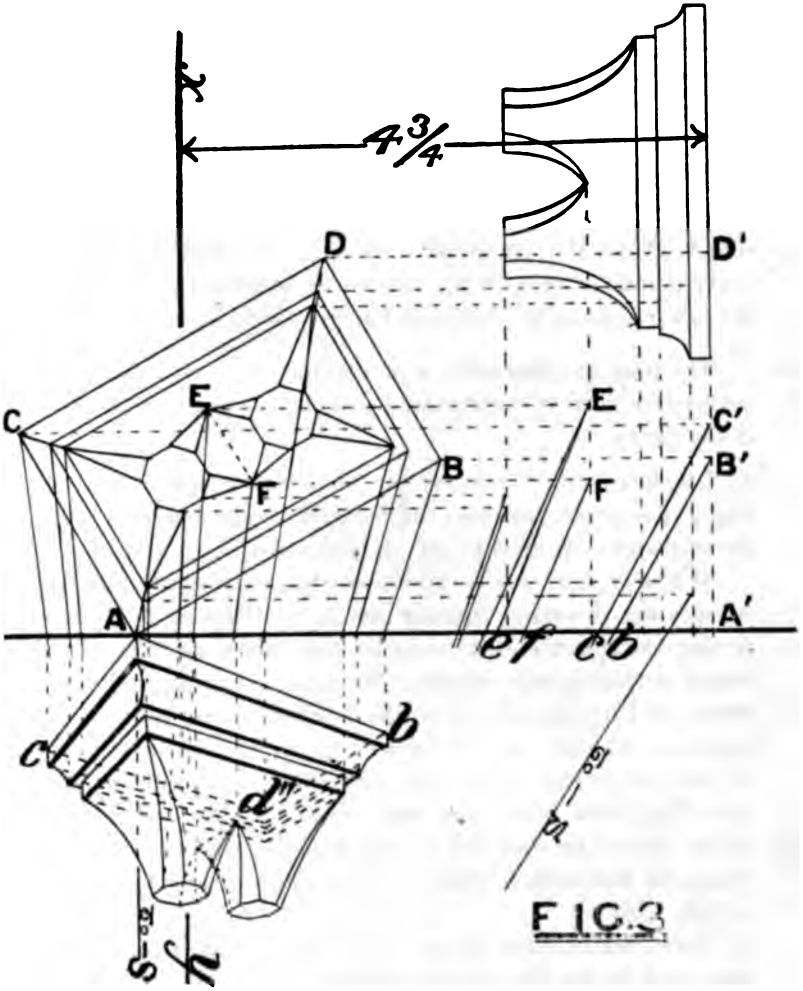


FIG. 3

are shown by lines proceeding from S to the angles of the object, cutting AA in points *a*, *b*, *c*, *d*, etc.[*]

Their perspective heights are obtained by taking *xy* at right angles to AA, and making an elevation of the solid at the given height above the horizon, and draw the visual rays to all angles of the solid (remembering that this view of the solid is a side elevation; the picture plane is therefore represented by the line *aefcb*) cutting the picture plane in points as *aefcb*, etc. Now, if we take the line AA as representing level of top of capital, we may transfer the distances of each point from top on the side view to the perspective at *cbd*, etc., and so on for every point. Points on the curves are obtained in precisely the same manner as shown in the preceding method.

---

[*] AA represents the plan of picture plane, and the points *a*, *b*, *c*, *d* are not shown.

No. 63, 1892. **The plan and elevation of a window are given. Make a perspective view of the window under the conditions laid down in the figure.**

The plan and elevation of the window, as shown in Fig. 1, are given, and also the horizon, the ground-line, and direction and distance to the station point.

We may here make use of the given horizon for our perspective drawing, thereby saving the labour of transferring the given elevation to another place, as it is much easier in this case to transfer the plan. Let us, therefore, draw the line GL parallel to the horizon in any convenient position, as shown, and the drawing may be proceeded with in the usual way, the only difference in this from the preceding ones being the semicircular arches. However, if the arches be enclosed by a rectangle, and the corners joined to the centre, points in the curve are obtained, as 1, 2, 3, 4, etc.

Now, all heights in any one plane, as A, B or C, are measured on the line of intersection of that plane with the picture plane, and then drawn to their respective vanishing points.

It will be seen that the heights of the smaller arches will be measured on the height line D, and also the back of the large arch-rib, which is in the same vertical plane as the smaller ones.

The outside will, of course, be measured on height-line E, and the corbels on line F.

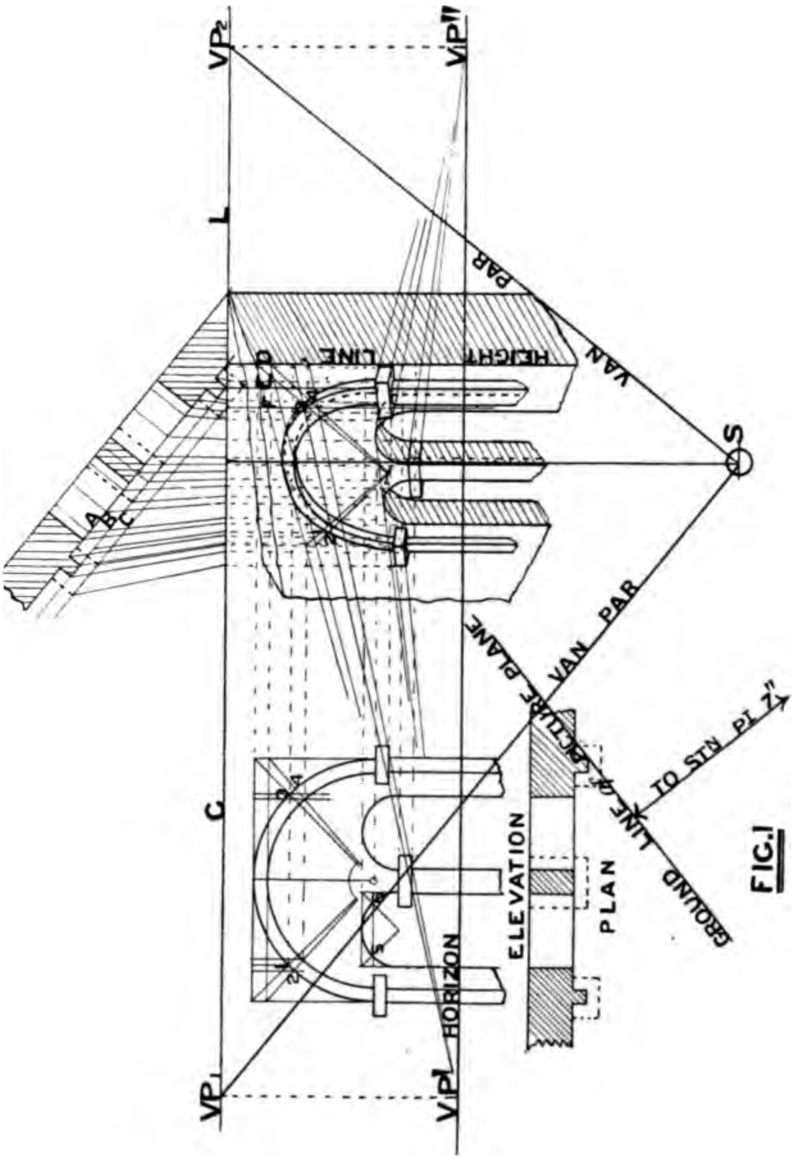

VP₂
V.P''
L
C
VP₁
V.P'
PAR
VAN
HEIGHT LINE
F C D
A B C
S
VAN PAR
PICTURE PLANE
LINE OF GROUND
TO STN PT 7"
HORIZON
ELEVATION
PLAN
1
2
3
4
5

FIG. I

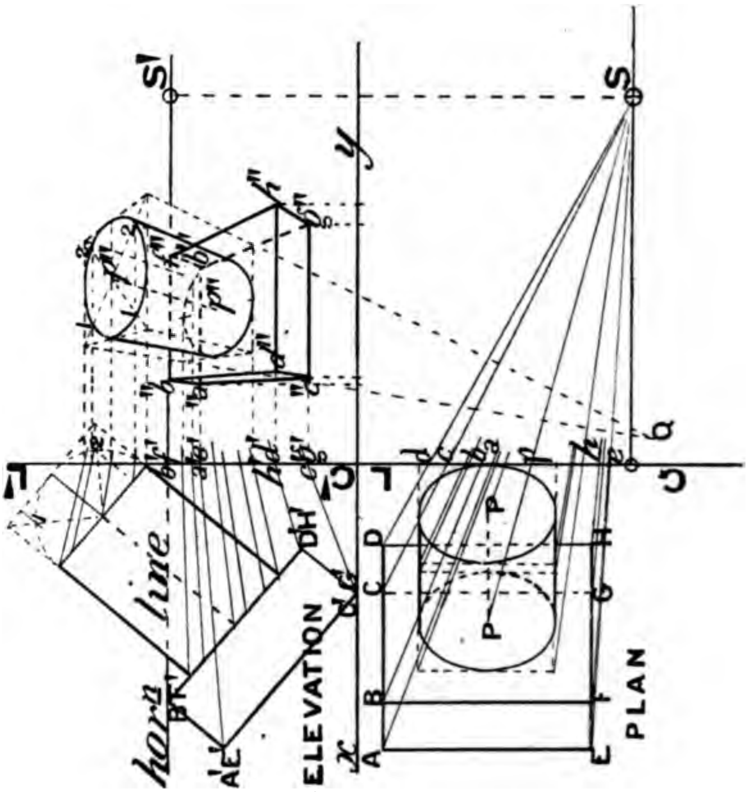
No. 63, 1893. **The diagram represents a cylinder resting on a block, tilted to an angle of 40°.**

Draw them in perspective. GL represents the traces of the picture plane, which touches the upper surface of the cylinder. The horizon line is on a level with the upper edge of the block.

The given figure is shown by its plan and elevation. Draw xy perpendicular to GL, and determine the elevation of the station-point, which is 4 in. in front of the picture plane.

Proceed as previously explained, and determine the plan and the elevation of the cone of rays emerging from point ss', and transfer the distances on GL, *i.e.* the perspective width of the object in plan to xy, and erect perpendiculars from each point to meet its ray in elevation.

The cylinder is enclosed by a square prism, points in the circles being obtained by the crossing-points of the diagonals, as at 1, 2. This needs no further explanation, as it is very simple, and follows immediately from the preceding problems.


S'
S
y
x
horn line
ELEVATION
PLAN
A
B
C
D
E
F
G
H
L
Q
P
AE
BF
DH

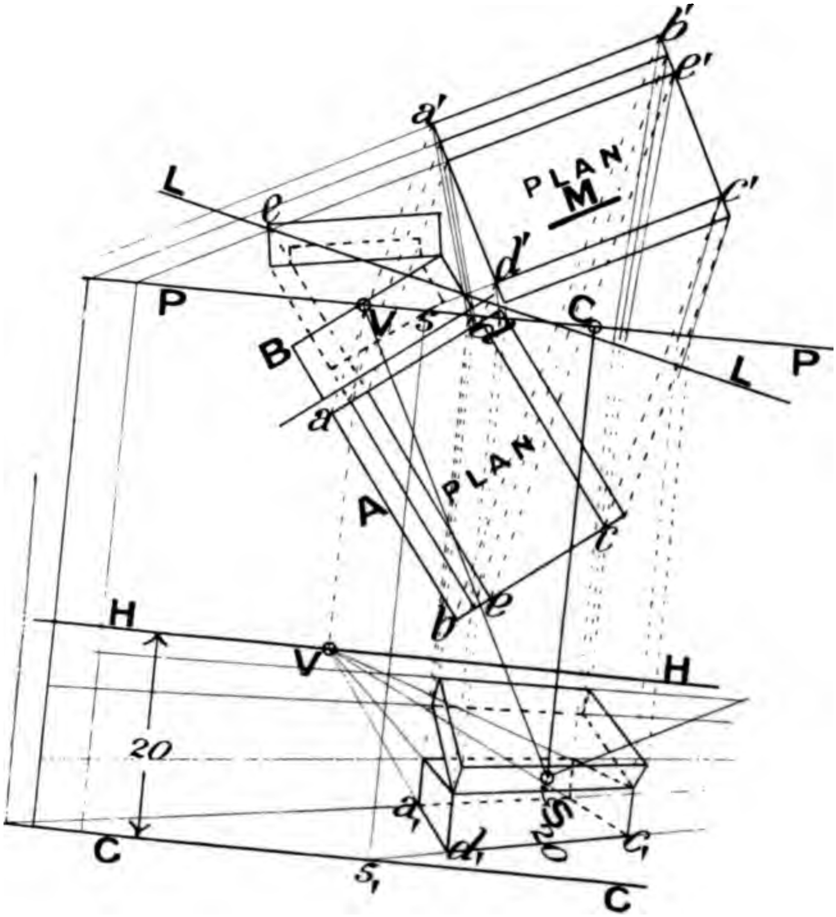
No. 63, 1894. **The plan and end elevation of an open box are given; LL is the ground-line of the plane of a vertical looking-glass. Taking the point S as the position of the eye, draw on the picture plane PP a view of the box as it would appear in the looking-glass. The angles of incidence and reflection of rays may be assumed equal. (Unit = 0·1 in.)**

(N.B.—In order to bring the figure within a small compass, the eye has been placed in such a position that the box would interfere with part of the reflection. This interference is to be neglected, and the complete reflection shown.)

The plan A, and the end elevation B, PP, LL, CS, and S20 are given.

The reflection is found by projecting every point or angle of the box perpendicular to the plane of the looking-glass LL, and measuring a distance behind LL on every projector equal to the distance that point is in front of LL. Thus a', b', e', c', d', etc., are as far behind LL as a, b, c, e, d, etc., are in front of it. All the corners being joined in order, the complete reflection is obtained.

Treat the image of the box $a'b'c'd'$ as the plan, and proceed as in previous problems. Line CC is taken as ground-line in any convenient position, and HH the horizon at 20 units above it. V, the vanishing-point to the left from S and parallel to $a'd'$, is projected back to V on HH. The vanishing-point to the right does not fall within the paper, but is found in precisely the same way.


b'
e'
a'
PLAN
M
L
e
d'
P
V
s
C
P
L
B
a
PLAN
A
c
b
e
H
V
H
20
a_1
d_1
S_0
c_1
C
s_1
C

No. 59, 1895. **The diagram gives the plan of a double-threaded screw lying on the horizontal plane. Draw the perspective of the outlines of the outer surface of the screw (*i.e.* on circumference *abc* in dotted end elevation) between the vertical planes indicated by the dotted lines AB, CD.**

PP is the ground-line (horizontal trace) of the picture plane; the sight point is at 3·65 in. from PP, opposite the point S; and the horizon-line is 4·60 in. above the ground-line.

We may commence by determining the station point S_1 3·65 in. in front of PP_1, and from it draw vanishing parallels to the parallel and perpendicular axis of the screw thread, cutting the PP in VP to the right and left of S respectively.

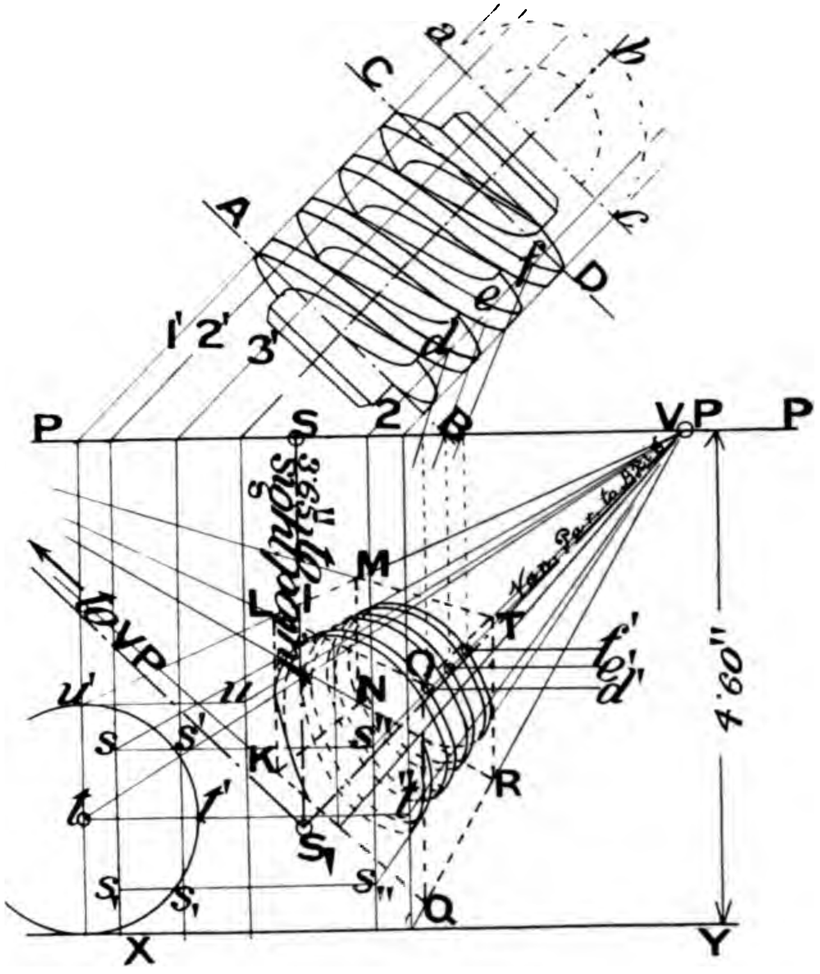
It will be found convenient to enclose the portion between the planes AB and CD by a square prism, and place it in perspective, as shown at KLMNOQRT.

For convenience PP is used as the horizontal line, and XY 4·60 in. below PP has been used as ground-line of picture plane.

Draw the circle *u's't'* with centre *t*, representing the end-view contour of the screw; and draw a series of vertical planes such as 1′, 2′, 3′ parallel to the axis of the screw, cutting each thread * above and below the axis; *e.g.* plane 2 on the right cuts one edge of the thread in *d*, *e*, and *f*. Produce this plane till it cuts the P.P., and project its trace in the picture plane perpendicular to PP. Now the heights of *d*, *e*, and *f* are seen in the end elevation at $S'S_1$. Project horizontally from $S'S_1$ to meet the trace of the plane in S'' and S_{11}. Also draw lines from $S''S_{11}$ to the V.P. (vanishing point) on the right. Then the lines $S''VP$ and $S_{11}VP$ are two lines which are in the same vertical plane, and lying on the screw, the upper line $S''VP$ passing through *d*, *e*, *f* in plan.

---

* It will be seen that a corresponding plane is on the left of the axis, with similar height.

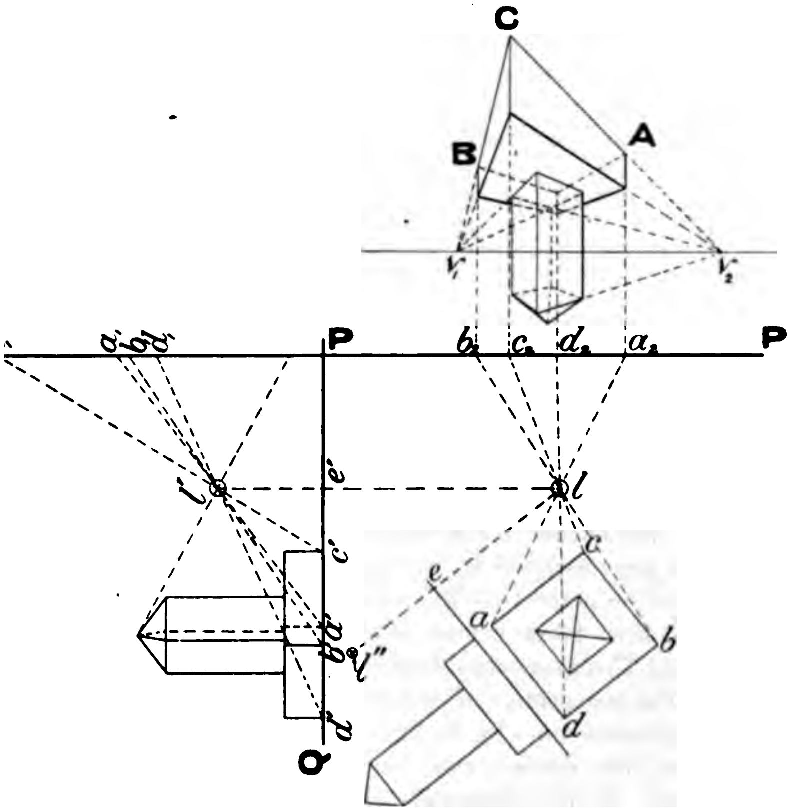

a
b
C
c
A
D
e
d
1' 2' 3'
2
B
P
S
V P
P
3.65"
M
L
T
f'
e'
d'
O
N
to VP
u'
u
s
s'
s'''
K
t
t'
t''
S'
s''
s,
s,
Q
R
X
Y
4.60"

Next draw the vanishing lines (in plan) from each point d, e, and f to the station point S_1, cutting PP in the points shown. Projectors from these points perpendicular to PP cut the line S″VP in points d', e', f', etc. Then these are points on the curve of the thread. Similarly proceed to find a number of other points. The student must be careful not to get the points so found mixed, otherwise he will find it very difficult to decide as to which of the threads they belong.

The method is precisely similar to the previous problems, and needs no further description. It will be noticed that the two-view method is unsuitable in this case, as the number of points is too great; and it would be extremely difficult to draw the elevation of the screw in the position given.

No. 62, 1894. **The plan and elevation of an obelisk are given. A photograph of this obelisk is taken on a vertical plane, of which PP is the ground-line. The lens is so placed that all the rays pass through the given point ll^1. Draw the resulting photograph.**

Take a new ground-line PQ perpendicular to PP, and draw the elevation of the obelisk and lens centre l on this ground-line. Draw the plans of rays of light proceeding from the four corners of the base, a, b, c, d, and passing through l, cutting PP in a_2, b_2, c_2, and d_2. At these points erect verticals: then the image of the points a, b, c, and d must lie in these verticals. The height of these points may be obtained by drawing the rays in elevation; then c_1P is the height of c above PP, and so on for all the points. The resulting figure is really an inverted perspective with ll' the station-point, and the process of drawing is facilitated by producing CB and CA and their parallel lines to find the vanishing points v_1, v_2.

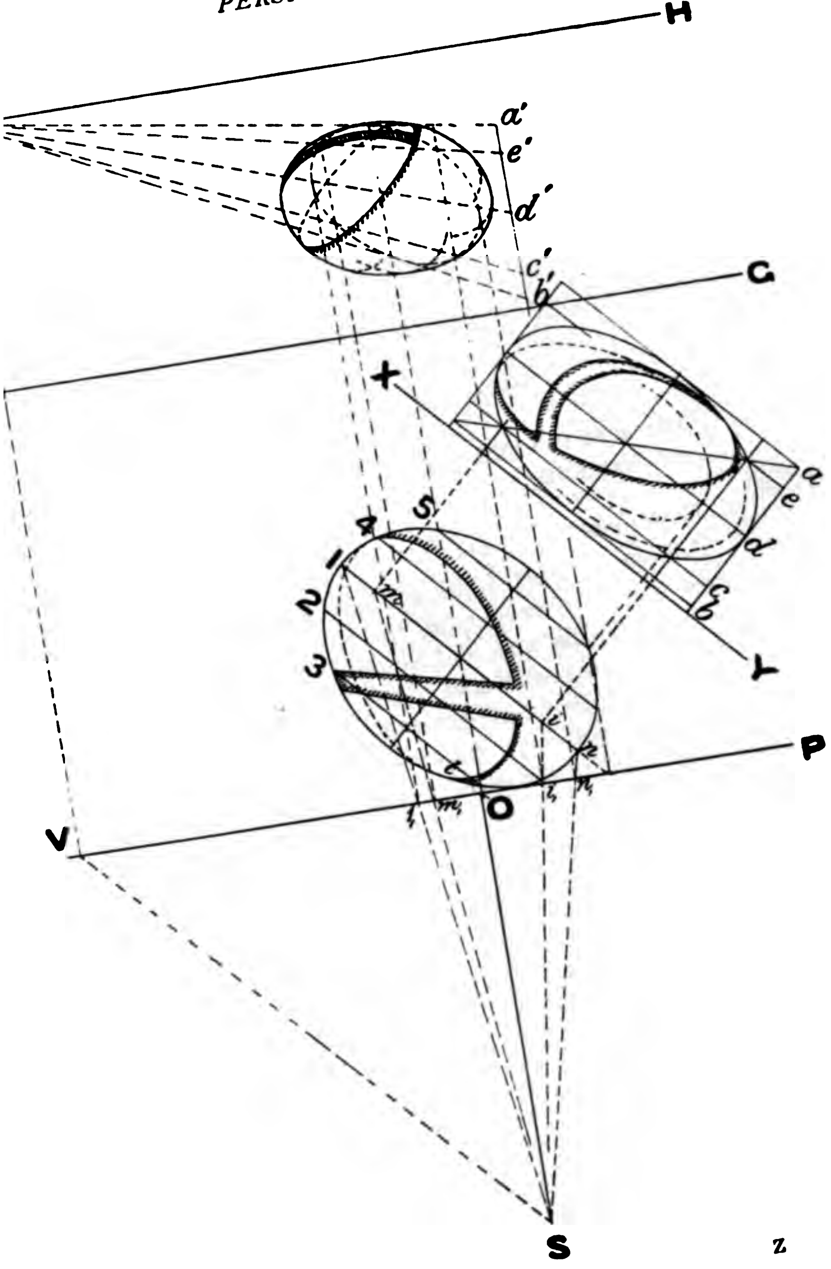

C
B
A
V_1
V_2
a_1
b_1
d_1
P
b_2
c_2
d_2
a_2
P
l'
e'
l
c'
e
c
a
b
b'
a'
l''
d'
d
Q

No. 59, 1896. **Draw the ellipsoid of the last question in perspectiv VP is the trace of the picture plane, 1*n* the plan of tl major axis. The dimensions are the same as in the la question. The sight point is on the line OS at S, such th OS = 5 in. The horizon is at the same height as the luminoı point in the last question. If the shadows are found l Question 58, and their perspective also determined, the enti marks given for Questions 58 and 59 will be 130.**

The plan and elevation of the ellipsoid with tł shadows have been obtained from the last question, an are shown in the accompanying figure. GG is the grounc line of the picture plane, and V′H the horizon.

The method of solution here adopted is to cut tł ellipsoid by a series of vertical planes parallel to the majc axis through 1, 2, 3, 4, and 5, and the outline which circum scribes the perspective of these sections will be the pe spective of the ellipsoid. It is not necessary to obtain th complete perspective of the sections, but only those part near the outline. The ellipses (sections of ellipsoid) hav been circumscribed by a rectangle in the usual manne It will be noted that the vertical sections of the ellipsoi have been taken symmetrically with regard to the majc axis. This saves considerable work.

The perspective of the shadow outline can be obtaine at the same time by finding the perspective of the point where the shadow cuts the ellipses, and joining up th points. In the perspective view only a very small portio of each of the section ellipses is shown. The student wh wishes to work this problem as an exercise should do i double the size given in the examination paper.


H
a'
e'
d'
c'
b'
G
X
a
e
d
c
b
Y
4
5
1
m
2
3
i
n
P
t
l
m,
O
i,
n,
V
S

CHAPTER IX.

MISCELLANEOUS EXAMPLES.

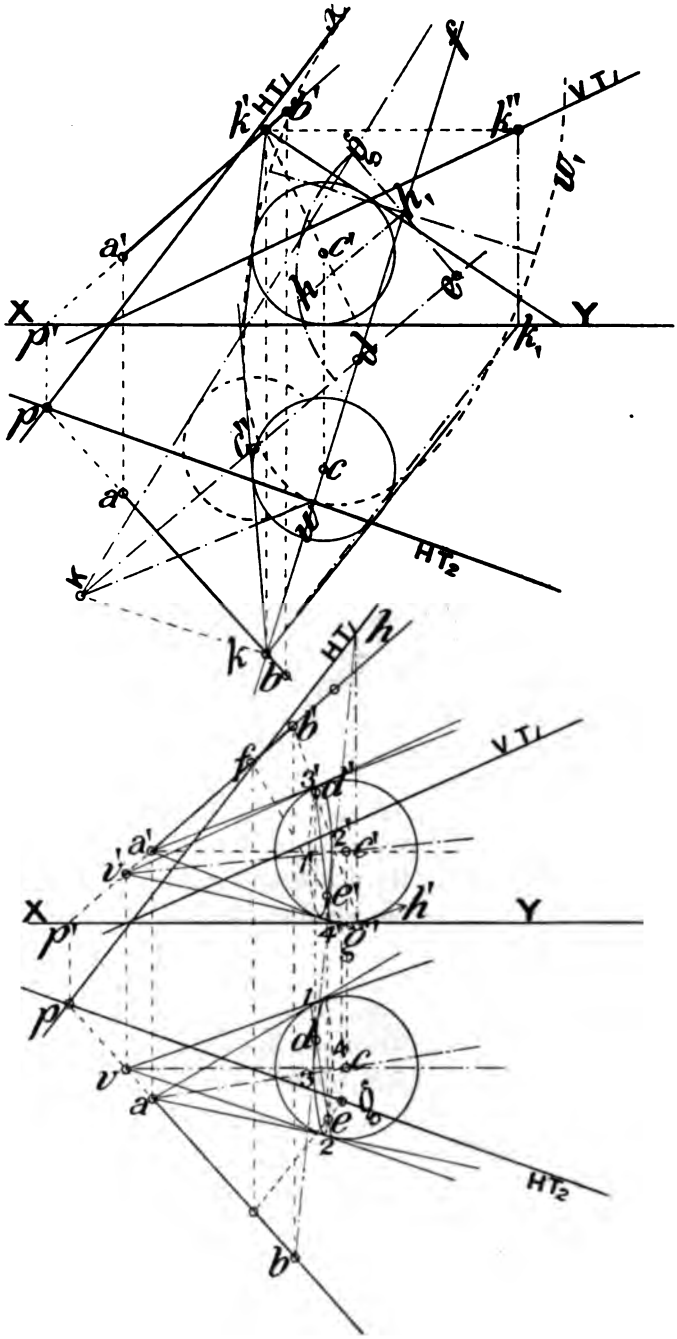
Determine two tangent planes to the given sphere, centre cc', and containing the line $aba'b'$.

First method.

Take any point k in the given line $a'b'ab$, as the vertex of a cone enveloping the sphere. Then the plane tangential to this cone and containing the given line will evidently be tangential also to the sphere.

Determine, therefore, the elliptic trace of the cone on the H.P., partly shown by the dotted ellipse xww_1, which is easily found if the axis kc be considered as xy, and determine the elevation of the cone as shown with its vertex at k'', centre of the sphere at c'', and axis $k''c''$, cutting kc, the ground-line, in point d. Bisect the distance between the points where the exterior generators cut the ground-line in point h, and through h_1 draw hge perpendicular to the axis $k''c''$, cutting the axis produced in point e and the exterior upper generator in point g. With e as centre, radius eg, describe the arc gh, and draw h_1h parallel to the axis; then h_1h will be the semi-minor axis of the elliptic base, and h_1w, the semi-major axis.

Through p, the horizontal trace of the line ab, draw the two lines px, pw tangential to the elliptic base of the cone. Then px and pw will be the horizontal traces


x
f
HT1
VT1
k'
b'
k''
w1
h1
a'
c'
h
e
X
Y
p'
k1
d
p
r
c
a
w
HT2
x
k
b
h
HT1
b
VT1
f
3'
d'
a'
v'
2'
c'
r
e'
h'
X
Y
p'
4'
g'
p
1
d
4
c
v
3
a
e
2
HT2
b

of the tangent planes to the sphere, and containing the line *ab*.

The vertical trace is found by drawing a level line on the plane HT_1 through any point, such as *k*, cutting the XY in point k_1, and erecting a perpendicular projector to cut the horizontal line through k' in point k''. Then k'' is a point in the vertical trace of the plane, and the crossing-point of HT with *xy* is another point; these, being joined, will be the vertical trace required.

The vertical trace of the other may be similarly found, but it falls outside the paper to the left in this case.

Second method.

Envelop the sphere with two cones; the axis of one, for convenience, may be taken horizontal, and the axis of the other may be taken parallel to the vertical plane, shown at *a'c'v'c'*, *acvc*.

The circle of contact of the cone axis *a'c'ac* is shown in elevation by the ellipse 1'2', and in plan by the straight line 1 2.

And the circle of contact of the cone axis *v'c'vc* is shown in elevation by the straight line 3'4', and in plan by the ellipse 3 4.

The crossing-points of these circles, *ede'd'*, will be contained by the tangent plane; or, in other words, they are the contact points of the planes and sphere.

Take any point, as *bb'*, in the line, and join to point *d'd*; then the plane required must contain the given line *a'b'ab* and the line *b'd'bd*.

Determine, therefore, the horizontal traces of *ab* and *bd* at *h* and *p*; join *hp*. Then the line *hp* will be the H.T. of the tangent plane to the sphere, in contact with it at point *d'd*.

The tangent plane to the sphere at point *ee'* is found in precisely the same way; *i.e.* find the horizontal trace of any line, as *fef'e'*, as shown at *g*. Then *pg* will be the horizontal trace of the tangent plane to the sphere, and in contact with it at point *ee'*.

The vertical trace is found as described in the first method.

Third method.

If a section of the sphere be made by a plane passing through its centre, and perpendicular to the given line *abdb'*, cutting it in point *hh'*, then the tangent lines to this section through *e* will evidently be lines contained by the required tangent planes.

Determine, therefore, a plane to pass through the sphere's centre and perpendicular to *ab*. To do this, take a new ground-line, *xy*, parallel to the line *ab*, and make a new elevation of the line and sphere as shown at *a''b''* and *c''*, the centre of sphere.

Through *c''* draw VT_3 perpendicular to *a''b''*, cutting it in point *e''*, and draw HT_3 perpendicular to *xy*.

Now rotate the plane about the trace HT_3 till it reaches the H.P., carrying the circular section of the sphere and point *e''* with it to f_1, G, and *e* respectively.

Through *e* draw the lines tangential to the dotted circle at points f_1 and G. These points of contact rotated back to the sphere will come to *f''* and *g''*, and their plans will be at *f* and *g*, also their elevations on the original elevation will be at *f'* and *g'*.

Then the required tangent planes will be found, as before, to contain the given line *ab*, and any line through *ab* and passing through the contact points *f*, *g*. Thus the horizontal trace of the line *ab* is at *p*, and that of the line *hg* is at *m*. Then the line *pm* will be the H.T. of the tangent plane to the sphere, and in contact with it at point *g*. Similarly, the horizontal trace of the line *bf* is at K. Then *p*K is the H.T. of the other tangent plane to the sphere, and in contact with it at point f_1.

The vertical traces will be found in the same way, as before explained.

Fourth method by Horizontal Projection (*unit* = 0·1 *in.*)

Let the index of the sphere's centre be $c12$, and the line ab, $a4$, $b40$.

Through c draw xy parallel to ab, and consider it as ground-line (working to level 12), and make an elevation of the line ab as shown at $a'b'$, *i.e.* point a will be 8 units below xy, or $12 - 4$, and b will be above xy 28 units, or $40 - 12$.

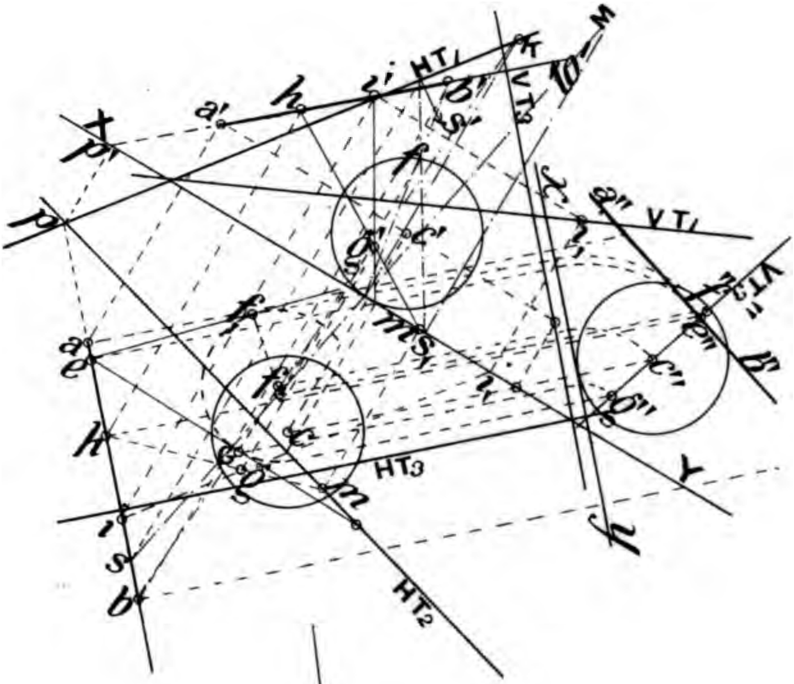
Next draw the vertical trace of a plane perpendicular to $a'b'$, and cutting the sphere through its centre, shown at VT, cutting the line $a'b'$ in point P. Now rotate the circular section of the sphere and point P to the H.P., when P will come to P_1, while the great circle will fall on precisely the same circle as the plan of the sphere (because we have taken xy through its centre, therefore the VT rotates about the centre c).

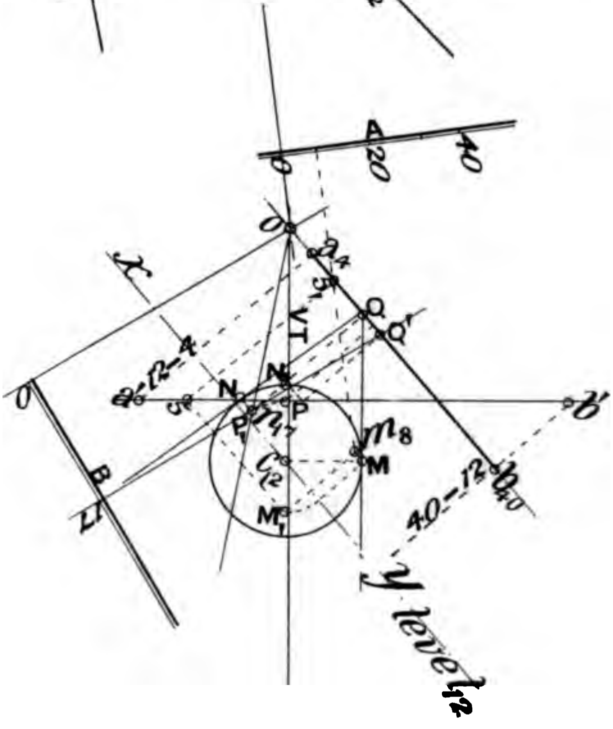
Project P_1 to Q on the line ab, and draw the lines QN, QM tangential to the rabatted circle, and in contact with it at points N and M.

Next through N and M back to the VT at N_1 and M_1, and determine their plans at n and m, indices of which are found to be 17 and 8 respectively.

The scale of slope of the tangent plane B is found by drawing the level line through N_1 to meet the line $a'b'$, and projecting back to Q′, when point Q′ will be on level 17, *i.e.* we found a point on the line $a'b'$ at the same height as point n. Then the line through $Q'n$ will be a level line on the tangent plane in contact with the sphere at point n. Draw the scale of slope at right angles with level line $Q'n$. Level o will be drawn parallel to $Q'n$ through level o of the line ab, as shown.

Similarly, the scale of slope of plane A may be found; *e.g.* draw the line M_15 parallel to xy, cutting $a'b'$ in point 5; erect the projector $5\ 5_1$ perpendicular to xy, cutting it in point 5_1. Then 5_1m8 will be level 8 on the tangent plane to the sphere, and in contact with it at point m. Level o will be drawn parallel to 5_1m8 through level o of line ab, when the scale of slope may be completed as shown.


a'
h'
i'
HT1
k'
b'
10
M
VT2
s'
x
p'
p
f'
c'
e'
m'
s
x1
a''
VT1
VT2
e''
c''
b''
a
e
f
c
h
e
m
HT3
HT2
i
s
b
Y
y


A
20
40
0
O
a4
5
Q
Q'
x
VT
12-4
N
N1
a
P
P1
n17
m8
M
c12
M1
b'
0
B
11
40-12
b40
y level 12

Determine two planes tangential to the given cylinder (axis $r'v'$), and inclined at θ° to the H.P.

Draw fgh, the traces of the plane of base of cylinder, and rotate this plane to the H.P., carrying the base of cylinder with it, centre R.

Through v' draw a line inclined at θ to the H.P., and consider it as a generator of a cone whose base angle is θ.

Now through Q, the H.T. of the axis of cylinder, draw Qm, Qm_1 tangential to the circular base of the cone, vertex v_1. Then Qm, Qm_1 are the traces of two planes inclined at θ°, and containing the axis of the given cylinder, cutting the H.T. of plane of base in points d and e. Join Rd, Re. Then Rd, Re are the lines of intersection of the planes Qm, Qm_1 with the plane of base of cylinder, after having been rotated back to its original position.

Draw Rb_1, Rb_2 at right angles to Re_1 and Rd respectively, when the points b_1 and b_2 will be the points of contact of the required planes with cylinder, and b_1c and b_2c_2 are the plans of the generators of contact.

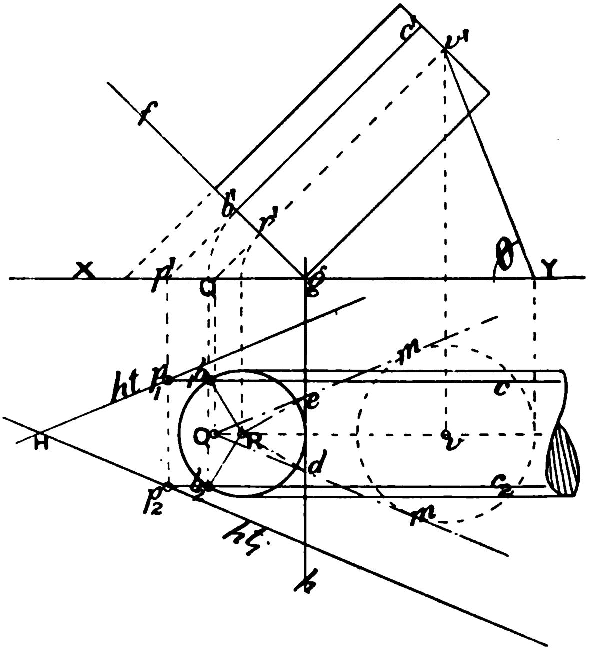
Now throw the plane of base of cylinder back to its original position, carrying the points b_1, b_2 with it to b'; draw $b'c'$ parallel to the axis of cylinder. Then $b'c'$ is the elevation of both generators of contact.

Determine their H.T.'s at p_1p_2. Through p_1p_2 draw the H.T.'s of the required planes parallel to the first traces already found, containing the axis of the cylinder, *e.g.* Qm, Qm_1.

Their elevations will be readily determined in the usual way, but are not shown on diagram.

The same result may be obtained if the cylinder were produced to the H.P., and its elliptic trace determined.

Then tangents to the elliptic base thus formed, and parallel to Qm, Qm_1, would be traces of the required planes.


d'
v'
f
b'
r'
X
p'
Q
g'
θ
Y
m
ht
p₁
p
e
c
H
Q
R
v
d
c₂
p₂
b₂
m
ht
h

Determine a plane tangential to the cone axis $v'd'$, an parallel to line $ab,a'b'$.

Draw the V.T. and H.T. of the plane of base of cone, a shown; and through the vertex vv' draw a line parallel t the given line $ab,a'b'$, and determine its H.T. at y, and th point where it intersects the plane of base at ee'.

Next rotate the plane of base about its H.T. to the H.P carrying the circular base of cone and the point e' with i to D and E.

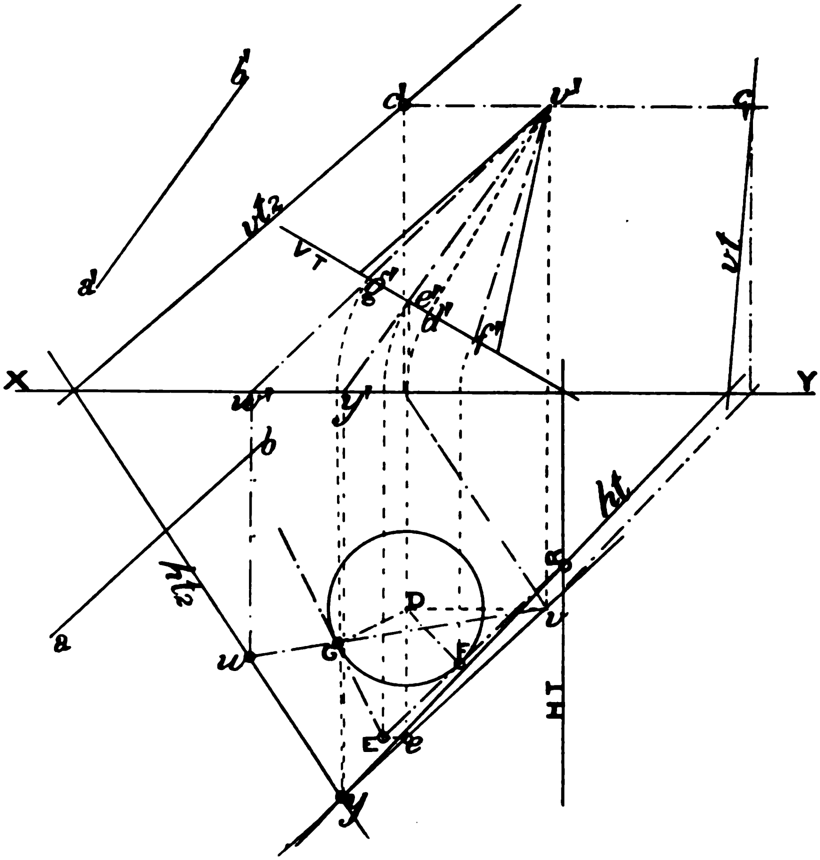
It will be seen that the cone, while in this position, ha its base on the H.P., and its axis vertical, while the point is also in the H.P. at E. Then through E draw two line tangential to the circular base of the cone at points G and F join DG, DF. Then DG, DF are the generators c contact.

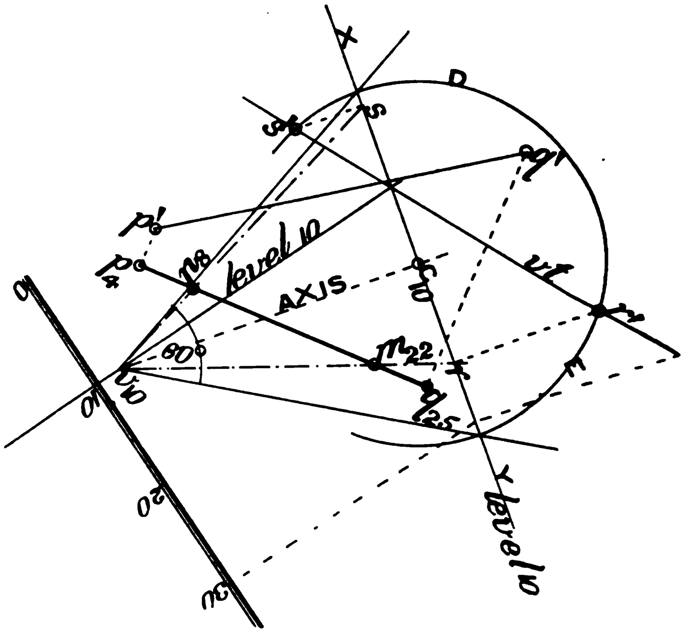
Now rotate the plane back to its original positio carrying points G and F with it to g', f'.

Determine the H.T. of the generator of contact $v'g'$ a u. Then the line yu is the H.T. of the plane tangential t the cone on the upper side. The H.T. of the generator v' is not found, but the tangent line EF is produced to cu the H.T. of plane of base in point R. Then R is in the H.T of the plane. Join yR. Then yR is the H.T. of the plan tangential to the cone on its lower surface.

Another method may be adopted, *i.e.* continue the con till it reaches the H.P., and draw its elliptic trace on the H.P and through the H.T. of the line ab draw tangents to th elliptic trace; these will be the H.T. of the required tangen planes.

The V.T. of the planes may be found readily by draw ing a level line through vv', the vertex of the cone, on eac plane, and finding their vertical traces at C′, C_1. The C′ C_1 are points in the vertical traces of the planes. Th crossing-points of the H.T.'s with the XY are also tw other points; these points joined will be the vertical trac respectively.


b'
c'
v'
q
a'
vt2
VT
g'
e'
d'
f'
vt
X
w'
y'
Y
b
ht
ht2
R
D
v
a
w
G
F
HT
E
e
y



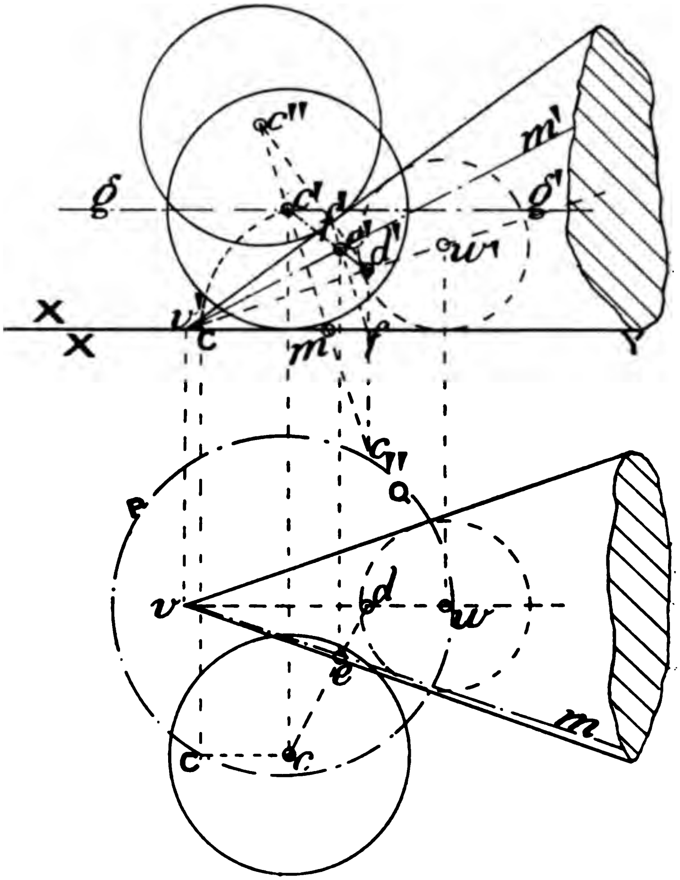
Determine the points of interpenetration of the line p with the cone, axis *v*C, and vertical angle of 60°, *p*4, *q*25, *v*10 C10.

Determine a plane to contain the given line and passin through the vertex of the cone. Then the crossing-point of the generators of the cone, cut by this plane and th line, will be the points required.

Determine the elevation of the line *pq*, as at *p'q'*, an find level 10, *i.e.* the same height as the vertex of the cone join this point to the vertex, and draw the scale of slop as shown. Level *o* is found by producing *p'q'* to cut th H.T., and draw a parallel to level 10, cutting scale of slop in point *o*.

Now draw XY anywhere at right angles to the axi and draw the circular base of the cone as at DE.

Draw the *vt* of the plane, cutting the base of the cone i *r's'*. Then the plans of the generators of contact will be *v* *vr*, cutting the given line in points *n* and *m*, the indices o which are 8 and 22. These are the required points of inter penetration.



A cone is given by its plan and elevation, axis *vw,v'w'*. Determine the position of a sphere of a given radius, in contact with it, at a given distance from the vertex.

Commence by drawing the elevation of the sphere, in contact with the cone at the given distance from the vertex, the centre of which is c''; the contact point is f.

Imagine this sphere to revolve round the cone till i reaches the H.P.; the path of the point f' will evidently b a circle on the cone, shown in elevation by the straigh line $f'f$. And the path of the centre c'' will also be a circle, shown in elevation at $c''c_{11}$. If the point $c''d'$, $c_{11}d$ be joined, a cone is formed, which is generated during rotation by the line $c''f'd'$ about the axis $v'w$.

Draw gg' parallel to XY, at a height equal to the radius of sphere, cutting the base of the imaginary cone in point c', then c' is the elevation of the centre of the sphere, when it has reached the H.P.

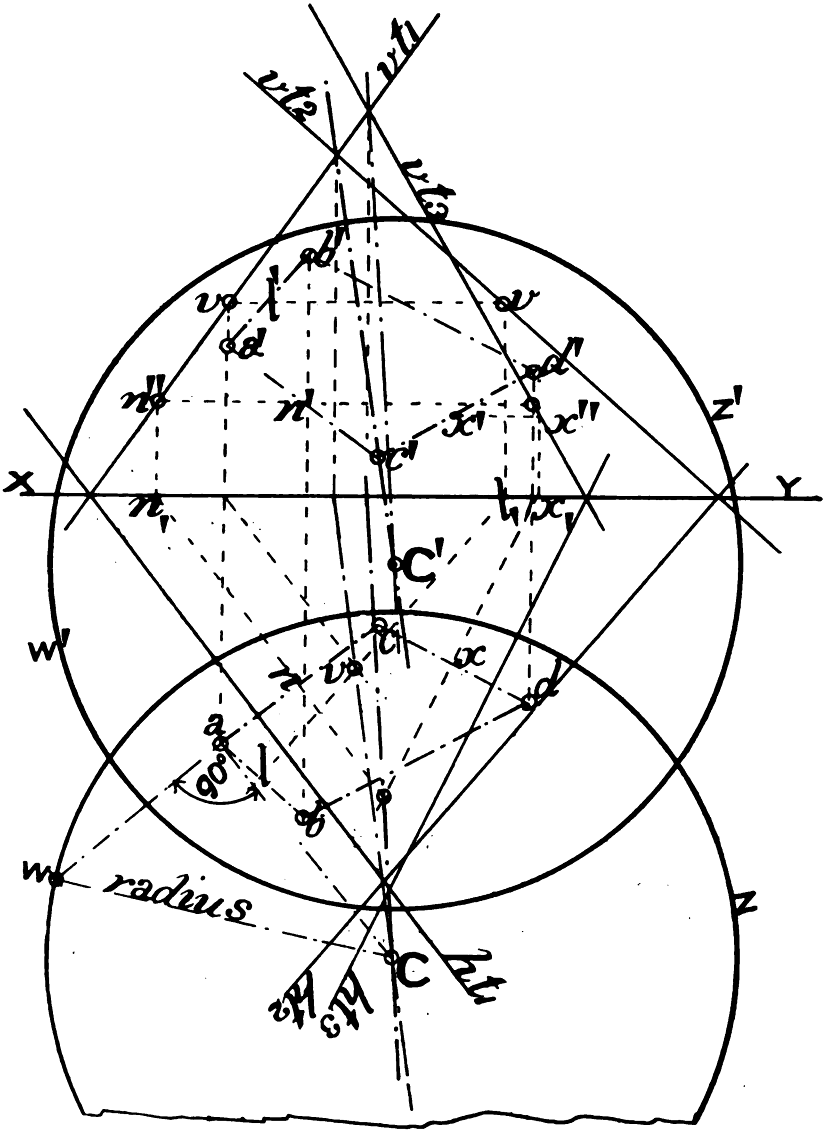
Join $d'c'$, cutting the circle of contact $f'f$ (of the point f' during rotation) in the point e'; then e' is the contact point of sphere and cone.

The plan of the sphere and point of contact will be readily obtained if the plane of the base of the cone be rotated to the H.P., as shown by the dotted circle PQ; the point c' will be on the circumference of the base at C. Now throw it back to its original position, when C will come to c; then c is the plan of the centre of the sphere. Join dc; then dc will be the plan length of $d'c''$. Project e' to e; then e is the plan of the contact point of sphere and cone; and $v'm'$ vm, are the plan and elevation of the generator of contact.

It will be seen that another sphere may be found on the other side of the cone.

Determine a sphere to pass through four given points ab, cd, $a'b'$, $c'd'$, no three of which shall be in one and the same straight line.

As the four given points will be contained by the surface of the sphere to be found, it is clear that if any two of the given points be joined, they will form a chord of a circle of the sphere. Hence the centre of the required sphere will be contained by the plane bisecting any of the chords, and the common point of intersection of any three of the planes bisecting any of the chords will be the centre of the sphere.



Commence, therefore, by determining the traces of the plane bisecting any chord, *aca'c'*. To do this, bisect *ac* in point *n*, and through *n* draw nn_1 perpendicular to *ac*, cutting XY in point n_1; at n_1 erect n_1n'', cutting a level line through *n'* in *n''*. Then through *n''* draw *vt*1 perpendicular to *a'c'*, and *ht*1 perpendicular to *ac*, when *vt*1, *ht*1 will be the traces of the plane bisecting the line or chord *aca'c'*.

Similarly, determine the traces of the planes bisecting the chords *ab*, *a'b'*, *cd*, *c'd'* respectively, at *vt*2, *ht*2, and *vt*3, *ht*3.

Determine their intersection as shown at C'C. Then C'C is the centre of the required sphere.

The radius will be readily found thus: Join C*a*; at *a* erect a perpendicular, *a*W; cut off a length *a*W equal to the difference of level of the points *a* and C; join WC. Then WC is the true length of *a*C, and is therefore the radius of the sphere containing the four given points.

Determine a sphere to pass through the three given points *a*, *b*, *d*, and touching the horizontal plane (unit = 0·1 in.).

Let the given points be indexed *a*4, *b*10, *d*20.

Commence by determining a plane to pass through the given points.* Thus join *ad*; at *a* and *d* erect perpendiculars equal to 4 and 20 units respectively. Let them be represented by aa^2, dd^2. Join a^2d^2, then a^2d^2 is the true length of the line *ad* (not shown). Now find a point in *ad* of level 10, and join to point *b*, which will be a level line at 10 units above the H.P. on the plane containing the points, and determine its scale of slope and its *ht* and *vt* as shown.

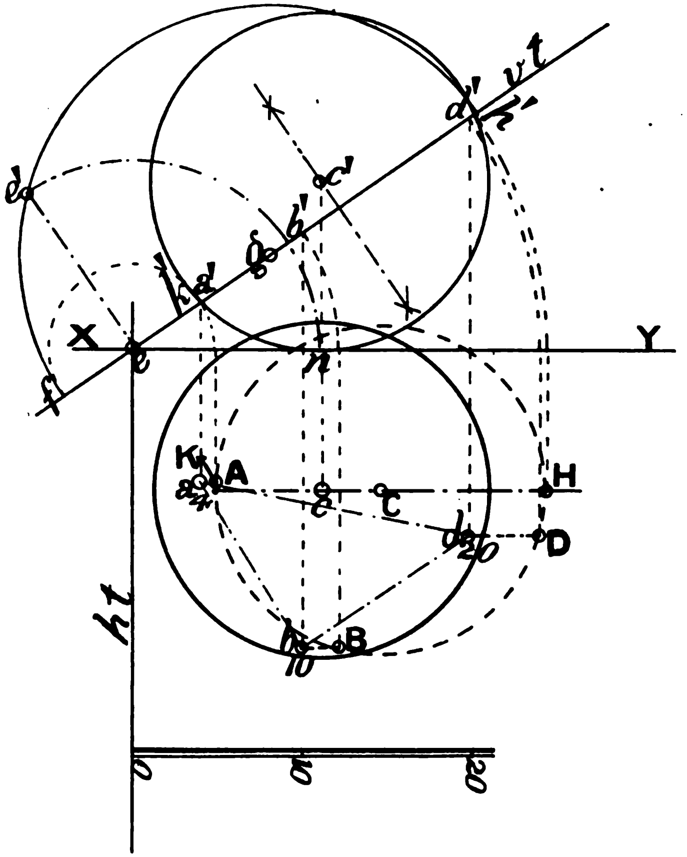
Project these points to *vt*, as at *a'b'd'*; now rotate the plane about its horizontal trace to the H.P., carrying the points *a'b'd'* with it to ABD.

Draw a circle through these points, having its centre at C, and draw the diameter through T, cutting the circumference of the dotted circle in K and H.

This circle will evidently be a small circle of the required sphere.

---

* The points must not be in one and the same straight line.



Now it is only necessary to determine a circle to pass through the two given points K and H, and to be tangential to the H.P. To do this, rotate the plane back to its original position, carrying the points K and H with it to *k'* and *h'*.

Determine the mean proportion between the lines *oh* and *ok'*, as shown at *oe'*; with *o* as centre, radius *oe'*, describe the arc, cutting XY in point *n*, at *n* erect a perpendicular to cut the line bisecting the diameter *k'h'* in *c'*.

Then c' is the centre of the required sphere, whose radiu is $c'n$. Determine the plan in the usual way as shown.

Note.—The latter part of this question resolves itse into the elementary question in plane geometry, of findin a circle to pass through two given points, and tangential t a given straight line; and the student should remembe that, if a line be drawn from any point o, cutting th circle in any chord $k'h'$, and if a tangent be also drawn t the circle from the same point such as on—

$$(on)^2 = oh \, . \, ok$$

or, in other words, the square on the tangent on is equal t the rectangle on oh, ok. Hence the tangent is equal to th mean proportional between oh and ok.

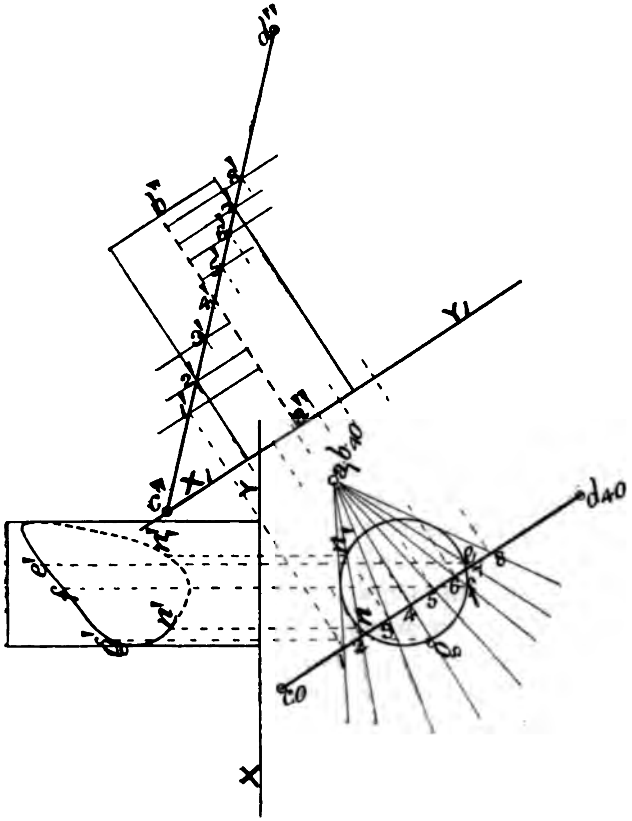
A skew surface is generated by a horizontal line movin in such a way as to be always touching the given lines abc a_1b_{40}, c_0d_{40}.

Determine the intersection of this surface with the vertic cylinder shown in plan and elevation (unit 0·1 in.).

Let the lines a_1b_{40}, c_0d_{40}, XY, and the plan of cylinde be given. Commence by drawing the ground-line X_1Y parallel to the line cd, and complete the elevation of th cylinder, lines cd and ab as at $a''b''$, $c''d''$.

Now through the plan of ab draw a series of radiatin lines, cutting the line cd in points 1 . . . 8. Determin their elevations at 1′ . . . 8′, each of which is of cours horizontal, and which also represents the height at whic the generating lines cut the cylinder.

The line, when in the position $a2$, cuts the cylinder i the points n, n_1. Through n, n_1 erect the projectors nn', n_1n perpendicular to the given XY, and equal in height to th height of the level 2 above X_1Y_1. Similarly, $a4$ cuts th cylinder in point g. Project through g to g', and make equa to the height of level 4 above X_1Y_1, and proceed in a simila way till a sufficient number of points be obtained; join b a freehand curve as shown, when the section is complete.

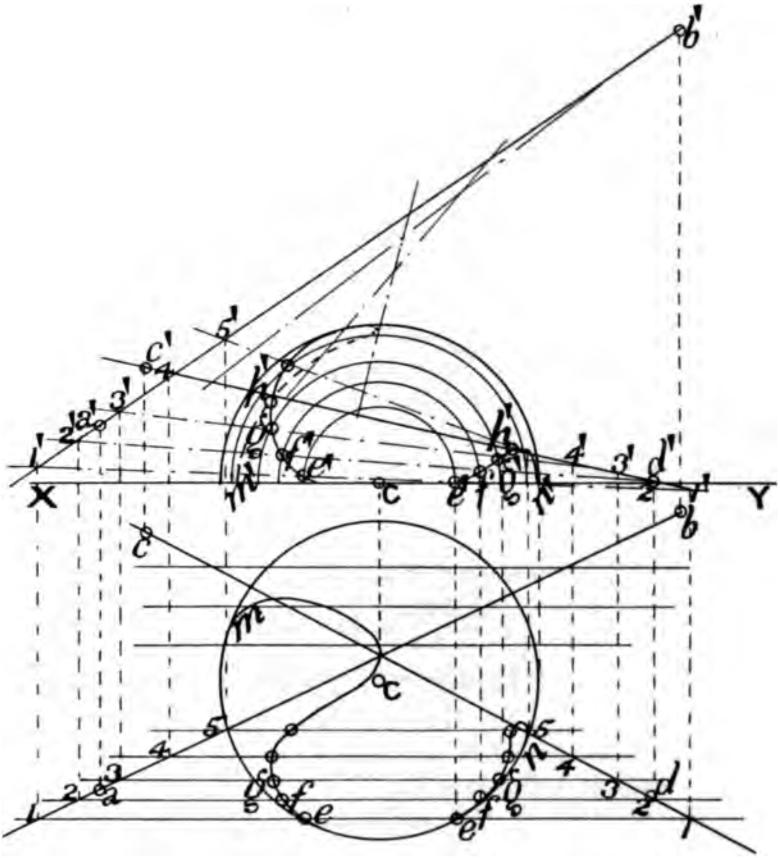

d''
b''
a''
1
2
3
4
5
6
7
8
Y1
X1
c''
Y
X
a₀b₄₀
d₄₀
c₀
e'
f'
g'
h'

A skew surface is generated by a line moving in suc way that it is always parallel to the vertical plane, and tou ing the given lines *ab*, *cd*, *a'b'*, *c'd'*.

Determine its intersection with the given hemispheri surface.

Take a series of vertical planes parallel to the giv vertical plane, cutting the hemisphere in semicircles, a the given lines in points.

The line joining these points will cut the semicircul sections thus obtained in the points of intersection of t skew surface with the hemispherical surface; *e.g.* consid the plane through the points 1, 1. Project 1, 1 to 1', 1' the lines *a'b'* and *c'd'* respectively; join 1', 1', cutting t first section of hemisphere in point *e* and *e'*. Similarly, t second plane cuts the lines in points 2, 2, 2', 2', line 2'2' cu the second circle in points *f*, *f*.' Proceed in the same w to *a*; find a number of points till the skew surface lea the hemispherical surface at the point m_1, these points bei joined by a freehand curve shown at *m*, *g*, *f*, *h*, *e*, on eith side of the centre *c*.



A bead slides with uniform acceleration down a generator of a right circular cone (radius of base 1·25 in., height 3·5 in.), starting from the vertex and reaching the base, while the generator traverses the surface of the cone (the generator revolves with uniform angular velocity round the axis of the cone) one and a half times.

Determine the developed locus of the bead on the cone.

As the acceleration is uniform, the space travelled over from the vertex will vary as the square of the time from starting, and as the squares of the ordinates of a parabola vary as their abscissæ measured from the vertex, a parabolic curve will represent the relation between the distance moved by the bead down a generator and the time of motion from the start.

Draw the plan and elevation of the cone as shown, and at $1'$ erect 1A perpendicular to the base of cone, and equal to the length of the generator.*

Next produce the line of base of cone to *c*, making 1C equal to one and a half times the circumference of the base of the cone *be*; divide the base into any number of equal divisions, as shown, 12 in this case. Now transfer these divisions to the line of base produced, as at 1_1, 2_1 3_1, . . . $12_1'$, 1_1, . . . C, or take one and a half times $2\pi r$, and measure it off from 1 to C; thus—

$$1\tfrac{1}{2} \times 2 \times \pi r = \tfrac{3}{2} \times 2 \times \tfrac{22}{7} \times \tfrac{5}{4}$$
$$= \frac{16 \cdot 5}{1 \cdot 4} = 11 \cdot 8 \text{ inches nearly}$$
$$(r = 1\tfrac{1}{4} \text{ in.}, \pi = 3\tfrac{1}{7} \text{ or } 3 \cdot 1416.)$$

Now at each of the divisions thus obtained erect perpendiculars to the XY as shown, and complete the parabola with vertex at A and axis A1.

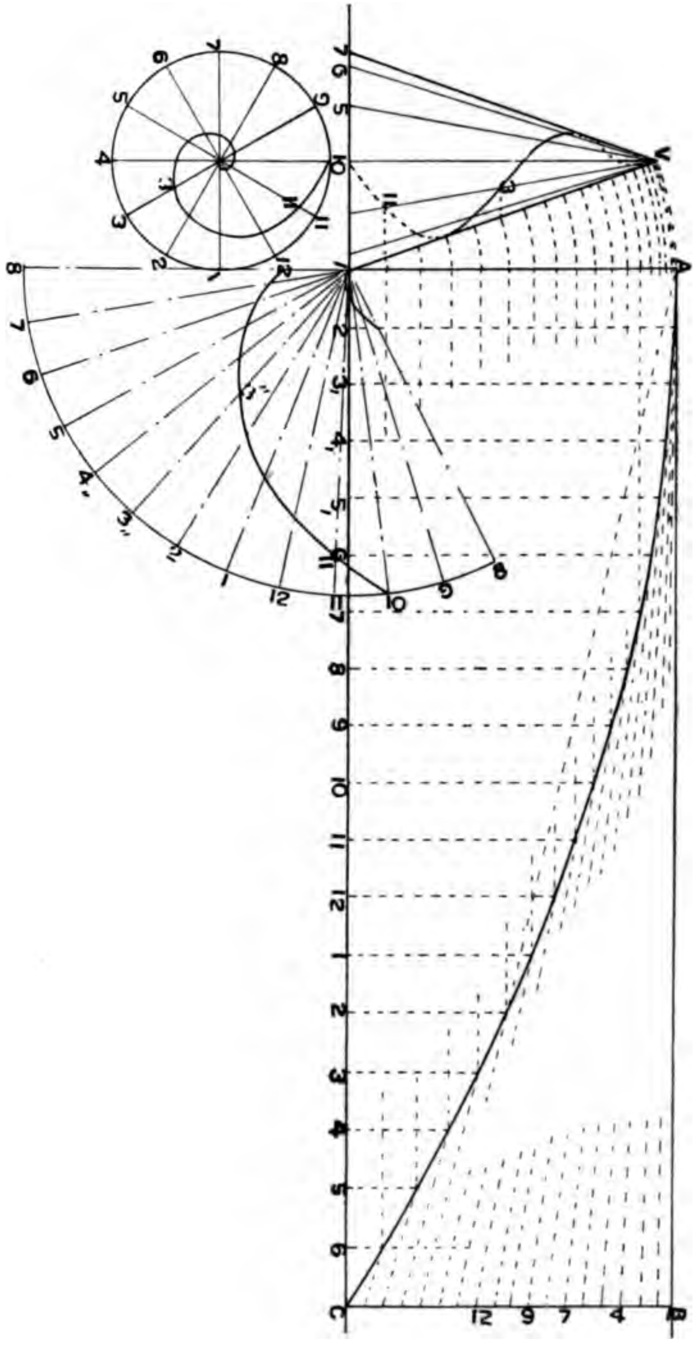
The abscissæ of this parabola, corresponding to the various points of division of 1C as ordinates, are the distances down the corresponding generators traversed by the bead.

Take, therefore, these lengths, and transfer them to their proper positions on the generator at its various stages of revolution; in the elevation it will be a helix curve, and in plan will be a spiral.

The developed surface of the cone is shown at $1_{11}8_{11}$, $3_{11}12_{11}8_{11}$, and the locus of the bead is there shown; thus $3_{11}3''$ on the developed surface corresponds to 3 3 on the generator and the elevation.

---

* See Eagle's "Descriptive Geometry," p. 257.

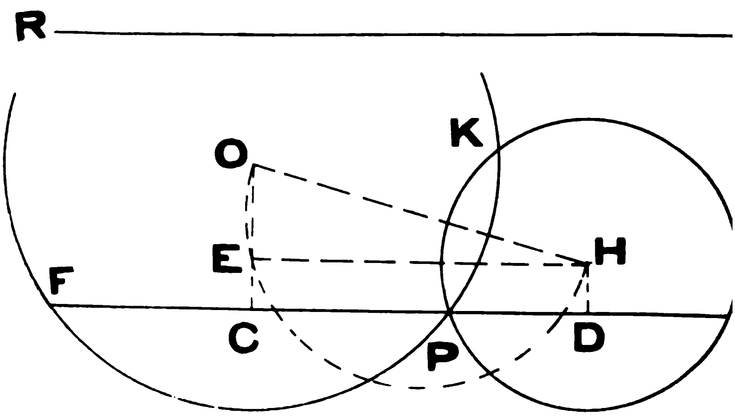

V
A
B
C
1
2
3
4
5
6
7
8
9
10
11
12
4,
3,,
2,
3,,
12
9
7
4

Through a point of intersection of two given circles, to dra a line whose length is given, its extremities terminating in t circumferences of the circles.

Let P be the intersection of the two circles, and RS t length of the given line. Describe a semicircle on OH diameter, OH being the line joining the centres of t circles. With H as centre, and radius equal to half the li RS, describe an arc cutting the semicircle OEH in Join HE, and through P draw a line parallel to HE, cutti the two circles in F and G; then FG is the required li equal to RS.

For if perpendiculars be dropped from the centres O a H on to FG, cutting it in C and D, then OC must pass throug E, because OEH is the angle in a semicircle, and therefore right angle. FP is bisected in C, and PG in D; therefo FG is twice CD. But CD = EH, and EH was made equ to half RS; therefore FG = RS.

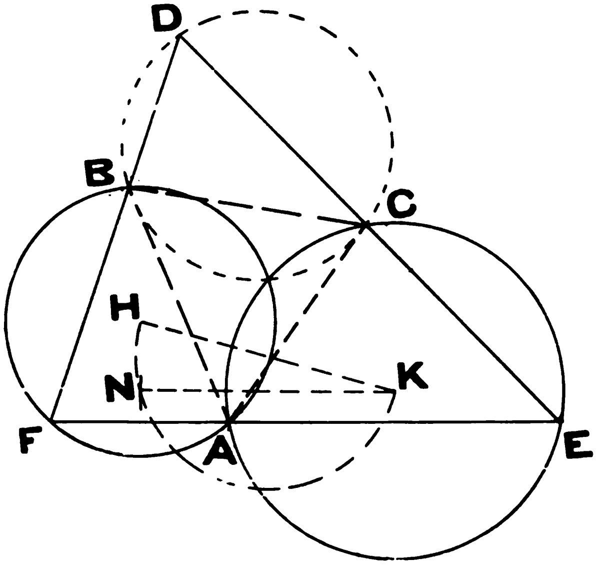
If it is required to put a line through P of maximu length, it must be drawn parallel to OH, because OH is t maximum length of EH, as EH is turned about H centre. The minimum line is PK.



Draw a triangle two angles of which are 72° and 46°, whose sides shall respectively pass through three given points, A, B, and C; and such that the two sides passing through A and B shall contain the angle of 72°, and the two sides passing through A and C shall contain the angle of 46°. Also the length of the side of the required triangle which bounds the angles of 72° and 46° shall be equal to 2½ in.

Join AB, BC, and CA. On BA as chord describe a circle whose segment shall contain the angle 72°; then this circle BFA will be the locus of the angular point F. In the same way, on AC as chord describe a circle whose segment shall contain an angle of 46°; then this circle CEA will be the locus of the angular point E.

Place in the two circles, BFA, CEA, a line whose length is equal to 2½ in., and which passes through A; thus, join HK and on it describe a semicircle. With K as centre and 1¼ in. as radius, describe an arc cutting the semicircle in N. Join NK, and through A draw FAE parallel to NK. Then by the previous problem FE = 2½ in. Join FB and produce it.



Join EC and produce it to cut FB in D; then DEF is t required triangle; for ∠ DFE = 72°, and ∠ DEF = 4 also FE = $2\frac{1}{2}$ in., and the three sides pass through A, B, a C as required.

It will be noticed that the intersection of FB and E must fall on the circle described on BC as chord, having angle in its segment equal to—

$$180° - 72° - 46° = 68°$$

the circle BDC being the only possible locus of the point when the angle BDC = 68°.

If, instead of the length of side FE being given, the ar of the triangle DEF is to be a *maximum*, FE must drawn through A parallel to HK. Because, if the area is be a maximum, the sides must be of maximum length, a the maximum length of FE is that drawn parallel to H (*vide* the end of last problem).

If the area is to be a *minimum* instead of a maximu then one side of the triangle to be drawn must fall upon o of the sides of ABC.

It also may be further noted that the construction pp. 360, 361 may be used to solve the following problen Given three points, draw a triangle whose sides shall pa through the three points, and equal the three sides of given triangle. This may be done provided the longe side of the given triangle does not exceed the greate distance between the centres of the circles, or shorter tha the shortest side of the triangle ABC.

Inscribe in a given triangle, DEF, another triangle, BA whose sides are equal to and similarly placed to those of t triangle XYZ, namely, the point Y on FE, the point X on D and the point Z on the side DF.

In the figure of the last problem, which is a forerunn of the present one, it will be noticed that the three circl through D, E, and F intersect in *one* point inside t triangle ABC. It will be necessary to prove that it is in every case. Using the present figure:* the circles BD and AFB intersect in O, and if O, is joined to A, B, and C, then $\angle$ AFB + $\angle$ AOB = 180°, they being opposite angles in a quadrilateral inscribed in a circle (Euc. III. 22). Similarly, $\angle$ BDC + $\angle$ BOC = 180°, $\therefore$ $\angle$ AFB + $\angle$ AOB + BDC + $\angle$ BOC = 360°. But 360° = $\angle$ AOB + $\angle$ BOC + $\angle$AOC. Therefore, after taking away the common parts AOB and BOC from each, we have $\angle$ AOC = $\angle$ AFB + $\angle$ BDC.

---

* See p. 365.

Add to each side of this equation the angle AEC; then—

$$\angle AOC + \angle AEC = \angle AFB + \angle BDC + \angle AEC$$

But the right-hand side equals 180°, therefore $\angle$ AOC + $\angle$ AEC = 180°, and hence (by Euc. III. 22) a circle can be circumscribed about the quadrilateral AOCE. Therefore the three circles do intersect in one point O.

Next, it may be shown that the angle FOE = $\angle$ BDC + $\angle$ BAC. Referring to the small figure in the right-hand top corner, the exterior angle fon (of the triangle fod) equals the sum of the two interior and opposite angles dfo and odf. Similarly, $\angle noe = \angle oed + \angle ode$. Adding these equals together, we have—

$$foe = fon + noe = ofd + fdo + ode + oed$$

But $foe = foa + aoe$; therefore—

$$foe = foa + aoe = fde + dfo + deo$$

Returning to the original diagram, we see with the same reasoning that—

$$\angle FOE = \angle FDE + \angle DFO + \angle DEO$$

But because the circles intersect in O, and that $\angle$ BFO and $\angle$ BAO are in the same segment on BO as chord, they must be equal. Similarly, $\angle$ CAO and $\angle$ CEO are equal; therefore $\angle$ FOE = $\angle$ BDC + $\angle$ BAO + $\angle$ CAO = $\angle$ BDC + $\angle$BAC, both of which are known before the problem is commenced.

Construction.—On FE as chord describe a circle, FO (dotted); have an angle in that segment of FOE = BD + XYZ. Also on DE as chord describe a segment, EO whose angle EOD = DFE + ZXY. These two segment intersect in O.

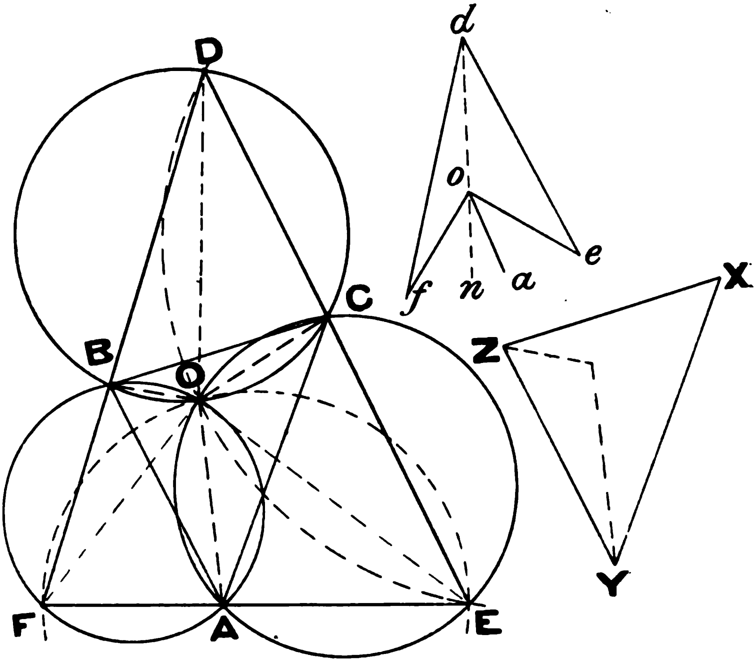
Because ∠ AFO and ∠ ABO are on the same chord A and in the same segment AFB, they are equal; therefor through Z draw ZP such that ∠ YZP = ∠ OBA = ∠ OF Similarly, ∠ OAB = ∠ OFB; hence make the angle ZY equal to BFO, the dotted lines intersecting in P. The because BA = ZY, the triangle ZPY is equal in ever respect to the triangle BOA.

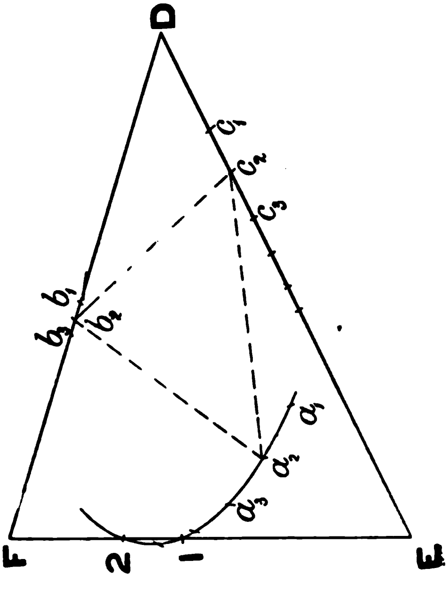
Therefore, with O as centre and radius YP, describe a arc cutting FE in A. Then with A as centre and ZY a radius, describe an arc, cutting DF in B. Further, with as centre and radius XY, describe an arc cutting DE in Join A, B, and C; then ABC is the required triangle.

If it is desired to inscribe in DEF a triangle, AB having given angles, and whose area shall be a *minimum*, it side BA must be a minimum, and therefore OA must be minimum, and that will happen when OA is perpendicula to FE.

NOTE.—The general problem above stated, of inscribin a triangle equal to another, is always possible as long as i is not less than the minimum triangle. There is n maximum triangle, as the sides DE, EF, and FD can b produced *ad infinitum*. This will be more readily see from the locus method, thus: Take any point c_1 in E and with that point as centre and XZ as radiu describe an arc cutting FD in b_1. With c_1 and b_1 as centre describe arcs with radii XY and YZ, cutting each other i a_1. In a similar manner obtain the points a_2, a_3, etc., in th locus of Y, as X and Z slide along ED and DF. Join thes points with a curve which cuts EF in two points 1 and Then the vertex A in the previous figure can be place at 1 or 2, and the triangle constructed.

If XY and YZ were very long, the curve a_1, a_2. a_3 woul


D
C
B
O
F
A
E
d
o
e
f
n
a
X
Z
Y


D
c_1
c_2
c_3
b_1
b_3
b_2
a_1
a_2
a_3
F
2
1
E

lie outside the triangle DEF and cut FE *produced* in t points, and however long they are they can always meet FE produced; hence there is no limit to the size of triangle XYZ, except a minimum.

Fig. P. 4. **NQMC is a quadrilateral. Find a point T whose lo is a semicircle on QN as diameter, such that when T is join to Q and N, the area of the figure MCNTQ is half as lar again as the original figure.**

Join CQ and bisect it in P. Join MN, and through draw SPR parallel to MN. Join MR and SN, either which bisects the quadrilateral QMCN. For, after joini MP and PN, the area of the $\triangle$ MPQ $= \triangle$ MPC, becau QC is bisected in P. Similarly, $\triangle$ QPN $= \triangle$ CPN. Ther fore $\triangle$ QMP $+ \triangle$ QPN $= \triangle$ MPC $+ \triangle$ CPN.

Also $\triangle$ SPM $= \triangle$ SNP, being on the same base ar between the same parallels, SR and MN. Further, the $\triangle$ SK is common to both, therefore $\triangle$ SKM $= \triangle$ KPN. To th half-area MPNC add the area MKS, and subtract the equ area KPN, the result being the figure CMSN; and to th half-figure MQNP take away the $\triangle$ MKS, and add th equal $\triangle$ KPN, the result being the $\triangle$ SNQ. Hence the lin SN bisects the quadrilateral MCNQ. On the other side QN set off a perpendicular length equal to the perpendicula distance of S from QN, and draw TW through the e tremity parallel with QN. On QN draw a semicircl cutting TW in T and W. Then either T or W will b the point required. Because the $\triangle$ QTN is on the sam base QN and between equal parallels to the $\triangle$ QSN, there fore the figure QMCNT is half as large again in area a QMCN.

P4

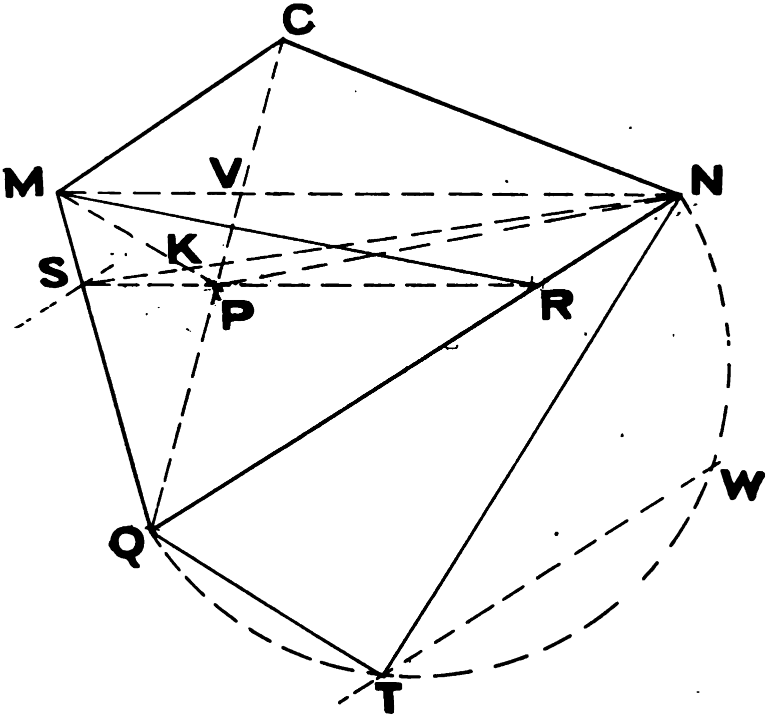
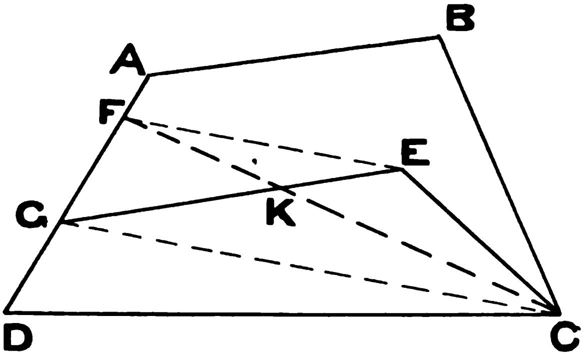


Fig. P. 5. **ABCD is the boundary of a field; CE is an existing hedge. Show where to place a hedge beginning at E, so that with CE it bisects the field ABCD. (No. 41, 1887.)**

Through the point C draw CF to bisect the area ABCD (see last problem).

Join FE, and through C draw CG parallel to FE. Join EG. Then CEG bisects the area of the field. For ABCD is bisected by CF, and the triangles GEC and GFC are on the same base and between the same parallels, and therefore equal. The triangle GKC is common to these two triangles, and therefore GKF = CEK. As these equals are on either side of CF, the former can be added to CFAB, and the latter taken away from it, and we have left CEGAB = CFAB = half the area of the field. Therefore the field is bisected by CEG.

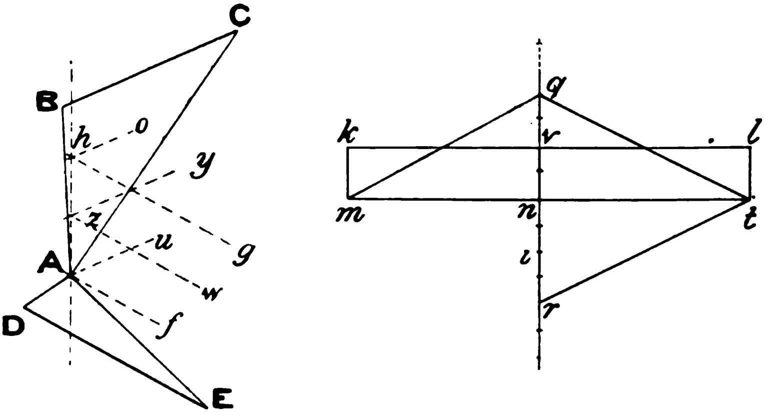
P 5



Determine the locus of the point A, such that, when joined to B, C, D, and E, the sum of the areas of the triangles ABC and ADE shall equal 1 sq. in. (No. 53, 1893.)

Draw the line *mnt*, and set off *mn* = BC, also *nt* = DE. On *mt* construct a rectangle, *mklt*, whose area equals the sum of the areas of the triangles ADE and ABC; that is, 1 sq. in. Set off the vertical height *vq* = *vn*, and draw *mq* and *qt*. Also set off *nr* = *nq*, and join *rt*. The area of the triangle *mqt* equals the area of the rectangle *mklt*; also, as *nr* = *nq*, the area *ntq* = area *ntr*; and consequently the area *mqn* + area *nrt* = 1 sq. in. Hence *qn* + *nr* = *qr* will under all circumstances be the sum of the heights of the two triangles ABC and ADE. Divide *qr* into any number of parts. Let *i* be one of the points of section, then *qi* is the height of the triangle on BC as base; and *ir* is the height of the triangle on DE as base. Draw A*f* parallel to DE, and at a distance *ir* from it. Also draw A*u* parallel to BC and *qi* from it. The intersection A of A*u* and A*f* is a point on the locus. In the same way, draw *zw* parallel to DE and *rn* away from it, and draw *zy* parallel to BC and *nq* away from it. The point *z*, the intersection of *zy* and *zw*, is another point on

P 6



the locus. Further, *h*, the intersection of *ho* and *hg*, is a point on the locus, when the distance between BC and *ho* is *vq*, and between *hg* and DE is *vr*. The locus *hz*A is found to be a straight line.

a. **A person in a train observes that the direction in which the rain appears to fall is inclined at 50° to the direction of the train's motion, which is at the rate of 45 miles per hour. The rain is really falling vertically. What is its velocity ?**

Also at another time the same person observes the inclination of the direction of the rain to be 50° when the train's velocity is 50 ft. per second; and further on, when its velocity is 30 ft. per second, the inclination is observed to be 70°. What was the actual velocity and direction of the rain ?

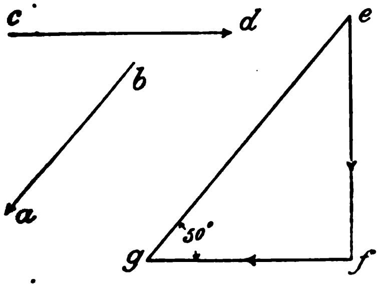
In the accompanying figure, *cd* is the direction of the train's motion, and *ba* that of the rain's apparent motion, inclined at 50° to *cd*. The apparent motion of the rain is its motion *relative* to that of the train, and to measure it we must reduce the train to rest by giving to the *train and rain* a velocity equal to that of the train in the negative direction. The rain's apparent velocity is then made up of two components—one of 66 ft. per second from right to left, and one of unknown magnitude vertically downwards. Set off *fg* to represent a negative velocity of 66 ft. per second as in the figure, putting in the arrow to denote direction; then through *f* draw a vertical *ef*, putting in an arrow here also to denote direction, and at the same time placing the foot of the vector *ef* at the point *f*, so that the two arrows have the same sense relative to *e* or *g* (if the rain's motion had been upwards, the line *ef* would have been below *f* instead of above it). Draw *eg* passing through *g* and making 50° with *fg*, cutting *ef* in *e*. Then *ef* represents the velocity of the rain, and scales off to be 72 ft. per second.

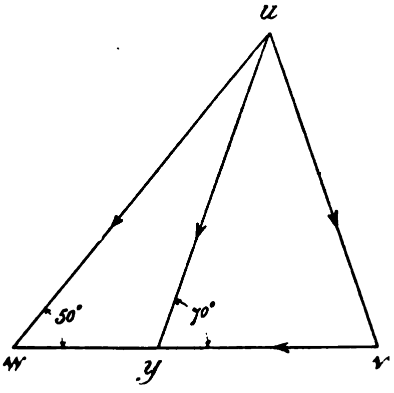
In the second case, the direction of the rain is not vertical, but inclined at a constant angle to the horizon.

In the figure let *uv* represent the real direction of the rain, and *vw* the negative velocity of the train at 50 ft. per second; then the *apparent* direction and velocity of the rain is given by *uw*.

Again, with *uv* as the real direction and velocity of the rain, and *vy* the negative velocity of train at 30 ft. per second, *uy* must be the then *apparent* direction of the rain. The intercept *wy* is the difference between the velocities of the train at the two different times, and equals 50 – 30 = 20 ft. per second.

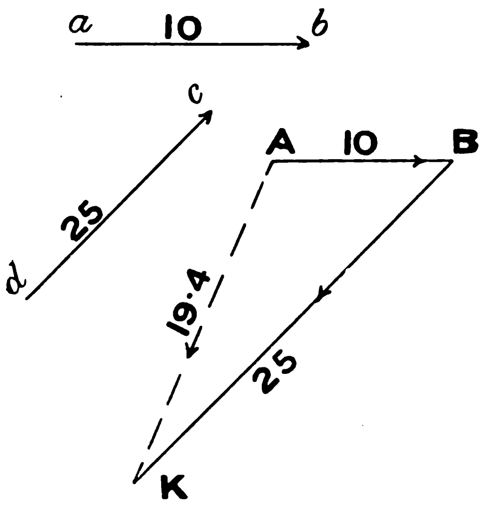
Set off the horizontal line *wy* to represent 20 ft. per second, and make the angle *vyu* 70°, and the angle *vwu* 50°, producing the lines till they meet in *u*. Set off *yv* to represent 30 ft. per second, and join *uv*. Then *uv* represents the direction and velocity of the rain, which scales off as 45·5 ft. per second.





β. **An ironclad is steaming at a rate of 10 knots an hour in the direction** *ab*, **while a torpedo-boat is steaming in the direction** *dc* **at the rate of 25 knots an hour. In what direction does the ironclad appear to be moving to the captain of the torpedo-boat, and with what apparent velocity?**

The relative positions of the two courses are given. If the torpedo-boat is reduced to rest, the velocity and direction of the ship can be measured. To reduce it to rest, give it a velocity equal to its own in the negative direction, and give the same velocity in the same direction to the ship also, because by treating them both alike, their *relative* conditions are not altered. After impressing on each the above negative velocity, the torpedo-boat is stationary, and the ship has a motion composed of two velocities—one of 10 knots per hour in the direction AB, and, in addition, another velocity of 25 knots in the direction BK. The resultant of these two velocities is represented by AK, which scales off 19·4 knots per hour. This is the velocity and direction in which the ironclad *appears* to be moving to a person on the torpedo-boat.



A kinematic chain or system of linkwork is arranged as in the accompanying figure, where C and D are fixed points, and CD = DF = 2 in.; also CG = CK = 6 in., and FG = GP = PK = KF = 3 in. Draw the locus of the point P.

After the point D was displaced along CD in either direction, what would then be the locus of P?

The points G and K of the hinged cell GFKP, on account of being controlled by the radius rods CG and CK, move in the same circle; and GK is always a chord. If DF were disconnected, and the points C and K made to approach towards or recede from one another, the points F and P must move along the other diagonal FP of the rhombus FP such that FQ = QP, and the diagonal FP produced must pass through C; for CK and KF are equal to CG and GF. Connect up DF; then F must move in the circle CFF_1. Hence draw any line CF_1P_1, cutting the circle CFF_1 in F_1. Draw two arcs intersecting in G_1, with radii CG and GF and centres C and F_1; also, with G_1 as centre and GP as radius, describe an arc cutting CF_1 in P_1: then P_1 is a point on the locus of P. Repeat the construction for a number of positions of CG (three will be enough), and join up the points. It will be found that they are in the same straight line. The result of this construction may be proved thus:

By Euc. I. 47—

$$CK^2 = CQ^2 + QK^2$$
$$\text{and } KP^2 = KQ^2 + QP^2$$

After subtraction, we get—

$$CK^2 - KP^2 = CQ^2 - PQ^2 = (CQ + QP)(CQ - QP)$$
$$= CP.CF = \text{a constant}$$
$$\text{because } CK^2 - KP^2 = \text{a constant}$$

But when CP.CF is constant for all positions of F, the locus of P is called the *inverse* of the locus of F.

Now the point F is made to move in a circle, and the inverse of a circle is a circle or a straight line (see Casey's "Sequel to Euclid," chapter on "Inversion"), the latter case being a circle of infinite radius. Referring to the smaller figure in the lower portion of the diagram, let b represent the length cd, and $x = ci$, also $y = fi$—that is, if c is taken as origin, and cn the axis of x; the co-ordinates of the point f are x and y. Further, let the co-ordinates of p be X and Y, with c as origin and the same axes. Then, from Euc. I. 47, we get—

$$di^2 + if^2 = df^2$$

that is, $(x-b)^2 + y^2 = a^2$, where a is the radius of the circle locus of f. The triangles cif and cnp are similar, and consequently—

$$\frac{ci}{cn} = \frac{if}{np}, \text{ that is } \frac{x}{\mathrm{X}} = \frac{y}{\mathrm{Y}}$$

Substituting the value of x, derived from this equation, in the preceding one, we get—

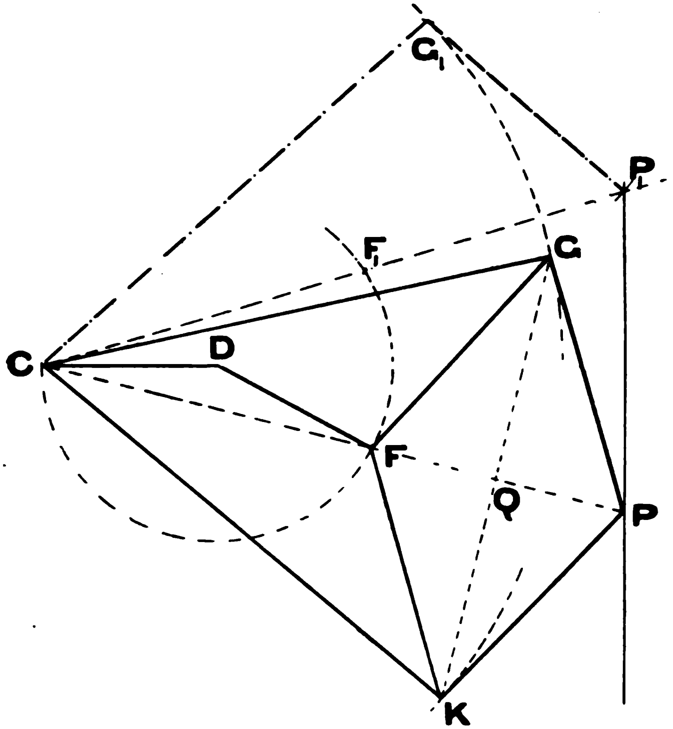
$$y^2\left(\frac{\mathrm{X}^2+\mathrm{Y}^2}{\mathrm{Y}^2}\right) - 2by\frac{\mathrm{X}}{\mathrm{Y}} + b^2 = a^2$$

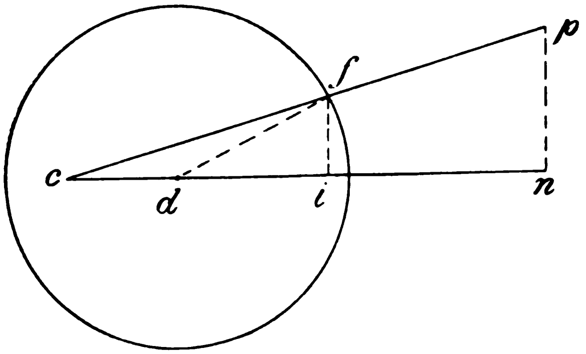
Again, because the locus of p is the *inverse* of the locus of f, then $cf \,.\, cp$ = a constant, say k^2. But $cf^2 = ci^2 + if^2$, and $cp^2 = cn^2 + np^2$; therefore—

$$k^2 = cf.\,cp = \sqrt{(x^2+y^2)(\mathrm{X}^2+\mathrm{Y}^2)}$$
$$\text{or } k^4 = (x^2+y^2)(\mathrm{X}^2+\mathrm{Y}^2)$$

Substituting the value of x derived from the equation $\frac{x}{\mathrm{X}} = \frac{y}{\mathrm{Y}}$ in this equation, we obtain—

$$y^2 = \frac{k^4\mathrm{Y}^2}{(\mathrm{X}^2+\mathrm{Y}^2)^2}$$


G₁
P₁
F₁
G
C
D
F
Q
P
K


p
f
c
d
i
n

Substituting this value of y in the equation—

$$y^2\left(\frac{X^2+Y^2}{Y^2}\right)-2by\frac{X}{Y}+b^2=a^2$$

we get—

$$X^2+\frac{2bk^2}{a^2-b^2}.X+Y^2=\frac{k^4}{a^2-b^2}$$

This is the equation to a circle,* therefore the locus of the inverse of a circle is a circle. If a is greater than b, the centre of the circle will be to the right of p; but if a is less than b, the centre will be to the left of p. When $a = b$, the last equation may be written as—

$$k^4 = X^2(a^2-b^2)+2bk^2X+Y^2(a^2-b^2)$$

obtained by multiplying through by a^2-b^2.

The coefficients of X^2 and Y^2 are now zero, and the equation becomes—

$$k^4 = 2bk^2X, \text{ or } X=\frac{k^2}{2b}=\frac{k^2}{2a}$$

which is the equation to a straight line perpendicular to the axis of X, situated at a distance $\frac{k^2}{2a}$ from the origin c. Hence the locus of P is a straight line.

---

* See any treatise on Analytical Geometry.

1. **A particle moves in such a way that at any instant the interval of time that has elapsed from the start is represented by the abscissa of a point on the given curve (to a scale of 4 seconds to the inch), while the distance travelled by the particle in that time is represented by the ordinate of the same point (to the scale of 10 ft. to the inch). Draw a diagram whose ordinate represents the velocity of the above particle, to a scale of 4 ft. per second to the inch.**

Also, if time were plotted vertically, and space horizontally, draw the curve, the scales being the same as above, except that the velocity can be represented to a scale of 0·5 ft. per second to the inch.

In the figure let ON be the time axis and lines perpendicular to ON represent the distances moved over at the end of the times represented by their corresponding abscissæ; for instance, the distance represented by NP to the scale of 10 ft. to the inch is moved over in a time represented by ON to a scale of 4 seconds to the inch.

Take two points, *b* and *a*, on the curve on either side of P; these points must be so close together that the line joining them is approximately the tangent at P, and such that their horizontal distance apart, *bc*, is approximately bisected by the ordinate PN. Then *ca* represents the increase in distance (or distance moved over) in the time represented by *bc*. In works on Mechanics, the distance moved over in a unit of time is called velocity, and the distance moved over per unit of time equals the distance moved over in any number of units of time divided by that number of units of time.* In the case in question the

$$\text{velocity at P} = \frac{\text{distance moved over in } bc \text{ units of time}}{bc \text{ units of time}}$$

$$= \frac{ac \times \text{scale of } ac}{bc \times \text{scale of } bc}$$

$$= \frac{\text{number of inches in } ac}{\text{number of inches in } bc} \times \frac{\text{number of feet to the inch}}{\text{number of seconds to the inch}}$$

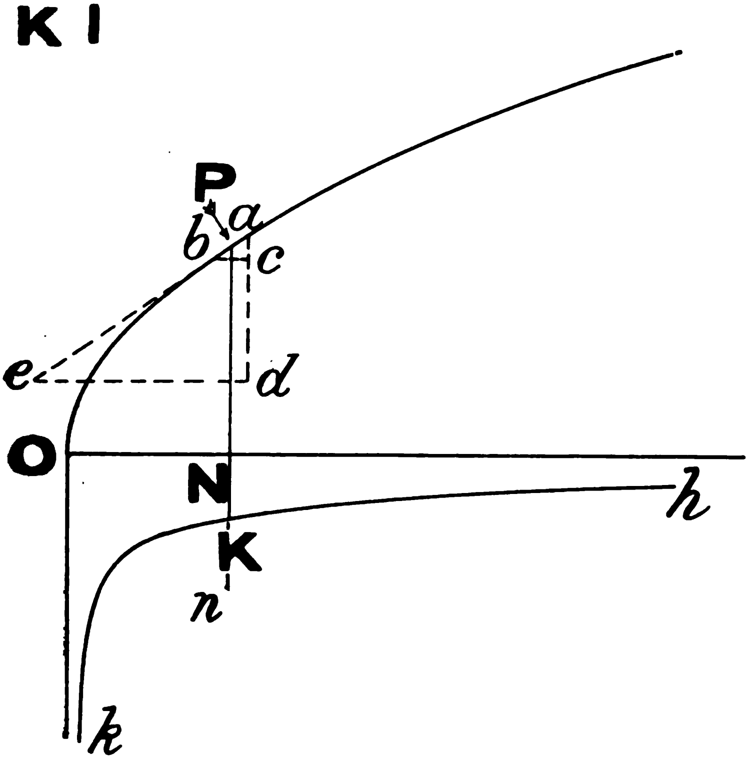
---

* It is here assumed that the velocity does not change appreciably during the interval of time. In the problem the intervals of time are selected so small that the velocity is practically constant during an interval.

But if we produce the tangent at P as far as necessary, and take any line *ed* parallel to *bc*, then, from similar triangles, $\frac{ac}{bc} = \frac{ad}{ed}$. Now, *ed* can be measured much more accurately than *bc*, so can *ad* than *ac*, and that is the reason for producing the tangent to some distance from P. Divide *ad* by *de*, and multiply the result by the ratio of the scales as shown above. Then plot the result downwards from N perpendicular to ON to the given scale of 4 ft. per second to the inch, thus: $\frac{ad}{de} = \frac{1 \text{ in.}}{1{\cdot}45 \text{ in.}}$, and the velocity at $P = \frac{1}{1{\cdot}45} \times \frac{10 \text{ ft. to the inch}}{4 \text{ secs. to the inch}} = 1{\cdot}73$ ft. per second, which, on a scale of 4 ft. per second to the inch, is represented by the ordinate $NK = \frac{1{\cdot}73}{4} = 0{\cdot}43$ in.

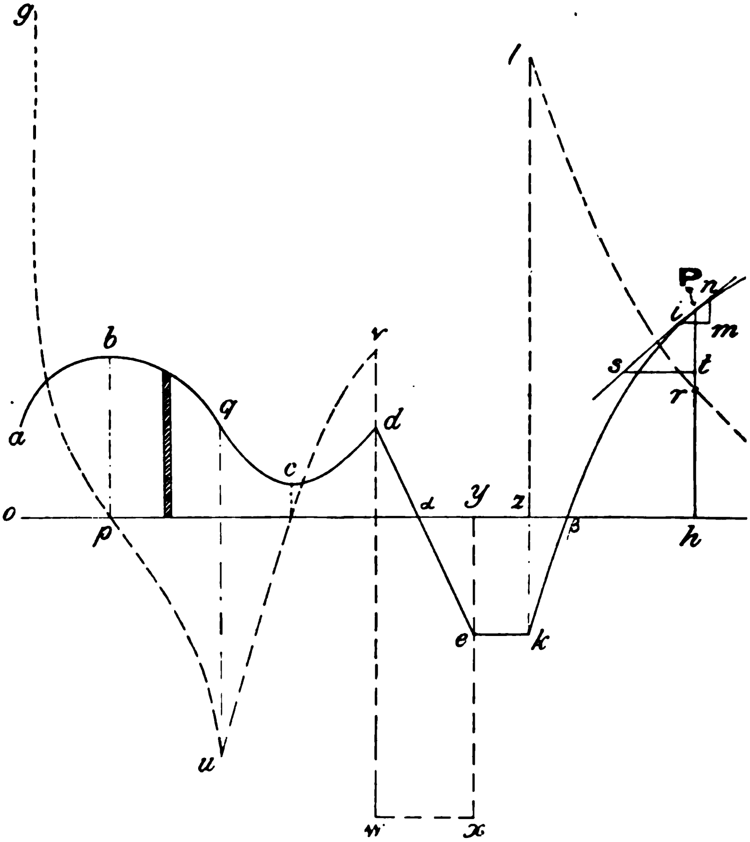
Repeating the construction for a number of points on the curve, we get the curve of velocity *k*K*h*.

If the times are represented by vertical ordinates and the distances by abscissæ, then the velocity at P = $\frac{\text{increase in distance}}{\text{increase in time}} = \frac{bc \times \text{scale of } bc}{ac \times \text{scale of } ac} = \frac{ed}{ad} \times \frac{4}{10} = 0{\cdot}58$ ft. per second, which to a scale of 0·5 ft. per second to the inch, is represented by $\frac{0{\cdot}58}{0{\cdot}5} = 1{\cdot}16$ in. Plot $Nn = 1{\cdot}16$ in., and repeat the construction for a number of points on the curve. The curve is not given, being left as an exercise for the student. The student is recommended to plot the curve on a time base and on a distance base, and to note the great difference in the curves.



K I
P
a
b
c
e
d
O
N
h
K
n
k

K 2



2. In the figure ordinates of the full-line curve represent velocities to a scale of 20 ft. per second to the inch, and abscissæ represent times to a scale of 5 seconds to the inch. Draw a curve of accelerations to a scale of 1 ft. per second to the inch.

Proceeding in the same manner as shown in the problem immediately preceding this one, take two points *i* and *n* on the curve equidistant from P, and so near together that that part of the curve approximately coincides with the tangent to the curve at P. Then, if *im* is horizontal, it represents the increase in time (or interval of time) during which the increase in velocity represented by *mn* takes place, and consequently—

$$\text{The acceleration} = \frac{\text{increase in velocity}}{\text{increase in time}}$$

$$= \frac{mn \times \text{scale of } mn}{im \times \text{scale of } im} = \frac{t\text{P}}{\text{S}t} \times \frac{20}{5}$$

Hence at a series of points, such as P, on the curve draw tangents, and then verticals through those points, such as P*t*, and finally any horizontals, such as *st*; then the acceleration at any point $= \dfrac{\text{vertical such as P}t}{\text{horizontal such as }st} \times \dfrac{\text{scale of P}t}{\text{scale of }st}$.

If *st* is taken of such a length that it represents 1 second, then *t*P will represent the actual acceleration.

Before attempting to plot the acceleration curve, it would be well to notice that the velocity increases between *a* and *b*, and therefore the acceleration will be positive between those points. Between *b* and *c* it decreases, and the acceleration is negative. From *e* to *k* it is constant, and there is no acceleration. At *b* and *c* the curve is horizontal, *i.e.* there is no increase or decrease there, and therefore the acceleration there must be zero.

It might also be noticed at this point that $\dfrac{\text{P}t}{ts}$ is the tangent of the inclination of the tangent (to the curve) to the horizon. When the tangent to the curve slopes upwards from left to right, the inclination lies between 0° and 90° and the tangent is therefore positive. Similarly, when it slopes upward from right to left the tangent of the inclination is negative, because the inclination lies between 90° and 180°. The dotted curve is the acceleration curve. At the point *q*, where there is a point of inflection, the curvature changes sign, and therefore there is a sudden change of direction of the acceleration curve below this point. The student who is familiar with the differential calculus will more readily appreciate these points. From *d* to *e* the rate of change of velocity is constant, and therefore the acceleration is constant. From *e* to *k* there is no change of velocity, and the acceleration is zero.

At the point *a*, where *de* cuts the base line, the velocity is zero, and consequently the body is at the instant at rest. The velocity is negative directly after, and consequently the body begins to move backwards until a time has elapsed represented by $\alpha\beta$. At β the body is again at rest, and directly after begins to move forward again.

As force = mass × acceleration, the force producing motion is represented by the ordinate to the acceleration curve. From *a* to *b* the force is pushing forwards, because it is positive. From *b* to *c* the force is pushing backward, and from *c* to *d* forward. At *d* the force suddenly changes from forward to backward, and is removed entirely between *e* and *k*. From *k* to *f* the body is pushed forward again.

It may further be noticed that the area of a small vertical strip of the velocity diagram, such as that shown shaded, is its width × average height; that is, the small interval of time × the average velocity during that time. This product is the distance moved over during the small interval of time, and consequently the area of the velocity curve gives the distance moved over. For example, the area between the base line *pa* and the curve *pbqca* represents the distance moved over during the time represented by *pa*. This area can be easily and quickly measured by a planimeter.

a. **1. A particle travels along the line OM. Its velocity at any instant is represented by the ordinate to the curve given, such as *np*.* Scales—horizontal, 10 ft. to the inch; vertical, 20 ft. per second to the inch. Draw a diagram showing the force at every point along the path OM.**

Take any *small* interval of distance *ek*, then at the beginning and end of that interval the velocities were *be* and *ak*. The gain in velocity during the interval was *ca*, and the distance travelled in that interval is *bc*. The acceleration of a body is the change of velocity per unit of time when the change is uniform, or if it is not uniform, it is the *small* increment of velocity divided by the corresponding *small* interval of time during which the change takes place, *i.e.*—

$$\text{Acceleration} = \frac{\text{increment of velocity}}{\text{increment of time}}$$

Multiply numerator and denominator by "increment of distance" and we get—

$$\text{Acceleration} = \frac{\text{increment of distance}}{\text{increment of time}} \times \frac{\text{increment of velocity}}{\text{increment of distance}}$$

The first fraction represents the velocity of the body, and the last fraction is $\frac{ac}{bc}$; therefore acceleration $= v \times \frac{ac}{bc}$ $= pn \times \frac{ac}{cb}$. Draw at p the normal *pm*; then the triangles *abc* and *pnm* are similar, and consequently $\frac{ac}{cb} = \frac{nm}{np}$.

---

* Unfortunately, *p* has been omitted from the diagram. It lies halfway between *a* and *b*. Also the points *e* and *k* lie at the intersection of *om* with verticals through *b* and *a*.

Substitute in the above equation for $\frac{ac}{cb} \times np$, and the acceleration $= nm$.

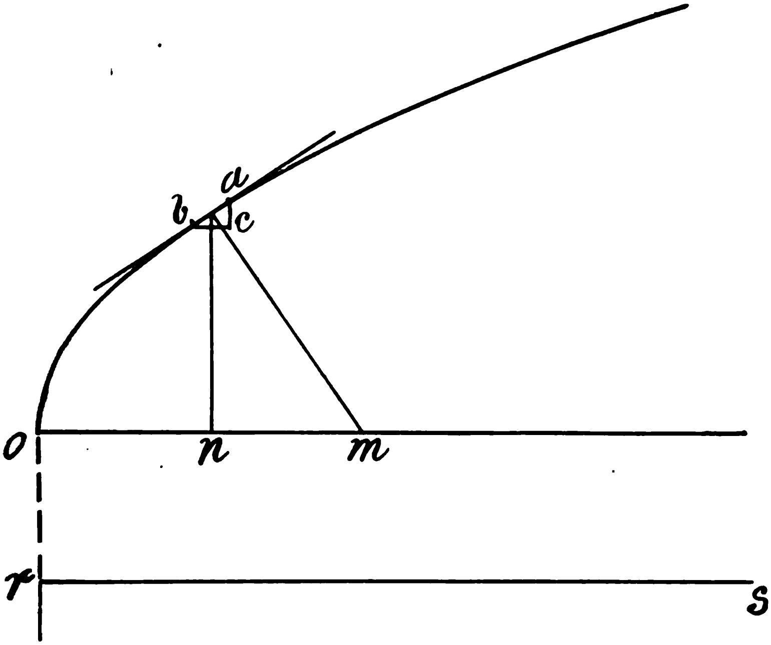
Hence, at a number of points along the curve draw normals cutting the axis *om*. Also draw ordinates cutting the same axis. The intercept cut off on the axis by the ordinate and normal represents the acceleration. This intercept, in the case of a parabola, is the subnormal, which is constant; therefore set off $or = nm$, and draw *rs* parallel to *om*: then *rs* is the acceleration curve. The actual acceleration is given by the length of the intercept in inches × scale of velocity squared ÷ scale of distance; for if in each of the equations we replace each of the quantities *ac*, *bc*, and *pn* by the actual quantities as scaled off the drawing—thus for *ac* write number of inches in *ac* × number of feet per second to the inch—the above result will be obtained.

Now, between the force acting on a body and the acceleration produced by that force there exists the relation, force = mass of body × acceleration produced by force. The mass is constant, and in the above case the acceleration is constant, and therefore the force acting on the body is constant. The curve of force will be similar to the curve of acceleration.

Fig. X. 4. **The line OP 2 in. long rotates about O as centre. PQRS is a rigid frame, of which Q always moves along the line OQ. Trace the loci of R and S during one complete revolution of OP. PQ = 5 in., QS = 2½ in., QR = 3½ in., the angle PQR is 130°, and the angle PQS is 90°. Also draw polar curves showing the velocity of R and S during one revolution when the angular velocity of the line OP is constant.**

Draw the crank OP in a great number of positions

K 3



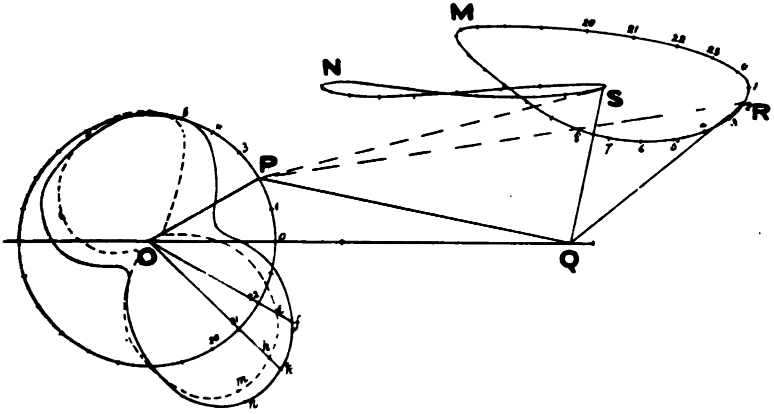
together with PQS and PQR, marking the positions of S and R for each position of OP. Joining these up, the curves RM and SN are obtained.

An accurate and very quick method of doing this is to trace the lines PQS and PQR on a piece of tracing-paper (although, of course, this cannot be done in an examination), and move PQ into a great number of positions, pricking through the points S and R for each position. This method will be found very useful in nearly all questions relating to loci.

To obtain the velocity curve, find points on the locus of R and the locus of S, corresponding to positions of OP, which are equidistant; thus divide the circumference of the circle into any number (in this case twenty-four) equal parts, and number the corresponding positions on the loci. Velocity, when constant, is the distance moved over in unit of time. Take as unit of time the interval during which OP is moving through one of the twenty-four angular divisions; then the distance moved over during one of those intervals (*i.e.* the velocity) will be the length of curve between two consecutive points, such as 20 and 21, or 21 and 22. As these lengths are comparatively small, they have all been multiplied by 4 in the polar diagram.

The lengths, such as between 21 and 22, represent approximately the average velocity of R between 21 and 22. This should be plotted on the arm OP from O when it is in a position midway between the points 21 and 22 on the circular path of P. But the points 21, 22, etc., are already found, and therefore it will save time if velocities can be plotted along the line OP when in the positions O21, O22, etc. The velocity of R at, say, 21 is half the distance between 21 and 22 *plus* half the distance between 20 and 21 (because that is the distance moved over in unit of time), *i.e.* half the distance between 20 and 22; and as the velocity is multiplied by 4 for the diagram, the diagram length for position 21 is the distance between 20 and 22

K 4



doubled. This length is plotted from O along O21, and is O*k*. The similar radius vector for the locus of S is O*h*. Proceed in a similar way for all positions of P, and the polar curves *nkf* and *mhe* are obtained for the velocities of R and S respectively.

Fig. K. 5. **The line OP (= 2 in.) rotates about the point O with a constant angular velocity of 100 turns per minute. The end Q of the line QP (= 5 in.) moves along the line OQ. Draw a diagram representing the velocity of Q during one complete turn of OP. Also draw a diagram representing the angular velocity of the line PQ during one turn of OP.**

The point Q, at the instant, is moving along QO at right angles to the vertical QW; therefore at the instant Q is turning about a point in the line (radius) QW. Similarly, the point P is instantaneously moving in the direction of the arrow perpendicular to OP, and therefore turning about some point in OPW at the instant. The line PQ must turn as a whole, at the instant, about some *one* point, which must be W. Let the instantaneous angular velocity of the $\triangle$ WQP about W be a; then, if v is the linear velocity of Q, and V the linear velocity of P, angular velocity × radius = linear velocity, or, $a \cdot WQ = v$, and $a \cdot WP = V$.

Dividing one by the other, we get—

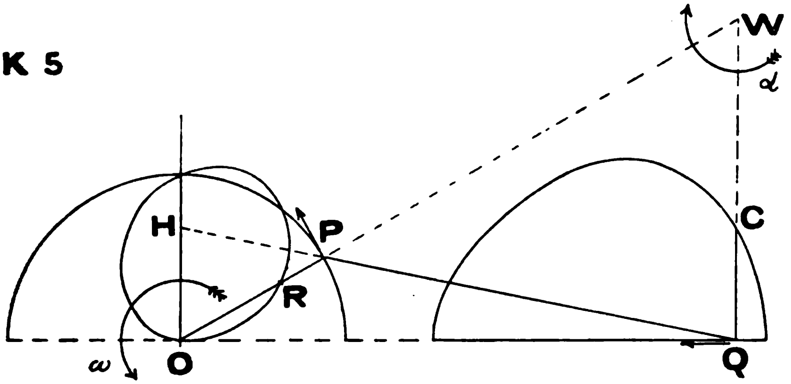
$$\frac{WQ}{WP} = \frac{v}{V}, \text{ or } v = V \cdot \frac{WQ}{WP}$$

Again, the point P also constantly turns about O with an angular velocity ω, and therefore $V = \omega \cdot OP$. Substituting in above, we obtain $v = \omega \cdot OP \times \frac{WQ}{WP}$. Produce QP to cut the vertical OH in H. Then, because OH is parallel to WQ, the triangles HOP and PQW are similar, and therefore $\frac{HO}{OP} = \frac{QW}{PW}$.

Substituting this in the above equation, we find that $v = \omega \cdot OH$.

Hence take a number of positions of P, and produce QP to cut the vertical in H; then OH × angular velocity of OP = velocity of Q. Plot OH above Q, such as CQ, and so on for a number of positions of Q, and the velocity curve

K 5



is obtained. The values of OH may be plotted on the arm OP, and we get the polar curve OR. The actual velocity in inches per second is obtained thus: The arm OP makes 100 turns per minute, or $\frac{10}{6}$ turns per second; that is, in one second it turns through an angle of $2\pi \times \frac{10}{6}$ radians. If OH is measured in inches, *i.e.* if CQ or OR is measured in inches, then $v = \text{OH} \times 2\pi\frac{10}{6}$ in. per second. If the figure had been reduced in scale to say one-eighth full size, then v would equal $\text{OH} \times 8 \times 2\pi\frac{10}{6}$ in. per second.

The angular velocity of the rod PQ is the same as that of the $\triangle$ WPQ, that is a. Dividing the previous equation $V = a \,.\, \text{WP}$ by the equation $V = \omega . \text{OP}$, we get—

$$a = \omega \cdot \frac{\text{OP}}{\text{WP}} = \omega \cdot \frac{\text{HP}}{\text{PQ}} \text{ from similar triangles}$$

Therefore plot lengths on the arm OP given by successive positions of HP, and the curve of angular velocity is obtained. The actual angular velocity in radians per second is—

$$\text{angular velocity } \omega \times \text{HP} \div \text{PQ}$$

The diagram is not given, but is left as an exercise. The figure represents the mechanism of the direct-acting engine.

Fig. X. 6. **In the accompanying figure P and Q are fixed points 3 in. apart, about which rotate the lines PC = 2 in. and QD = 3 in. The ends C and D are connected by a line CD of constant length, 4 in. Draw a diagram of the angular velocity of the arm DQ, when the arm CP rotates at a uniform rate of 20 turns per minute.**

Also, if CD were 2 in. long, and PQ 1 in., draw the angular velocity diagram of DQ. Also show a method of obtaining the angular velocity of CD.

In the small figure of the accompanying diagram, let BC be the line of centres, while DE is the connecting-link of the two cranks BD and CE. Produce the cranks to intersect in A, and the connecting-link DE in both directions. From B, A, and C drop perpendiculars on to DE produced, and let DE cut BC in F. From similar triangles we obtain the following ratios:—

$$\frac{AD}{BD} = \frac{AY}{BX}, \frac{CE}{EA} = \frac{CZ}{AY}, \text{ and } \frac{BF}{CF} = \frac{BX}{CZ}$$

Multiply all the left-hand sides together and all the right-hand sides together, and then—

$$\frac{AD \times CE \times BF}{BD \times EA \times CF} = \frac{AY \times CZ \times BX}{BX \times AY \times CZ}$$

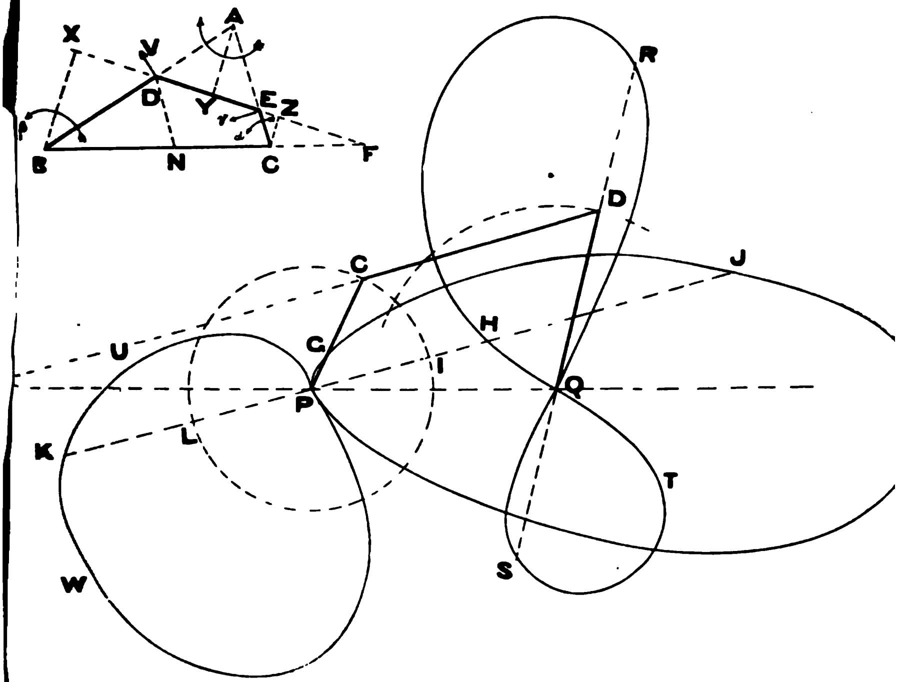
But the right-hand fraction is unity, as all the factors cancel; therefore—

$$AD \times CE \times BF = BD \times EA \times CF$$

In other words, the products of the alternate segments into which the transversal DE divides the sides of the triangle ABC are equal.

Let δ be the angular velocity with which the triangle ADE instantaneously turns about A; then, if V is the linear velocity of D at the instant, and v the linear velocity of E, $V = \delta . AD$, and $v = \delta . AE$. Also by division—

$$\frac{v}{V} = \frac{AE}{AD}$$



But if α is the angular velocity of EC at the instant, and β the angular velocity of BD, then $V = \beta . BD$, and $v = \alpha . EC$. Also—

$$\frac{v}{V} = \frac{\alpha . EC}{\beta . BD}$$

Substituting this value of $\frac{v}{V}$ in the above equation, we have—

$$\frac{\alpha . EC}{\beta . BD} = \frac{AE}{AD}, \text{ or } \frac{\alpha . EC . AD}{\beta . BD . AE} = 1$$

But it has been shown above that—

$$\frac{AD . CE . BF}{BD . EA . CF} = 1$$

therefore—

$$\frac{\alpha . EC . AD}{\beta . BD . AE} = \frac{AD . CE . BF}{BD . EA . CF}$$

Cancelling out equal factors, we have—

$$\frac{\alpha}{\beta} = \frac{BF}{CF}$$

or the angular velocities of the two cranks are inversely as the segments into which the line of centres BC is divided by the connecting-link DE, produced if necessary.

Hence draw the mechanism in several different positions, and in each of them produce the connecting-link to cut the line of centres. Measure off the segments, and calculate their ratio. Plot this ratio along the arm QD from the centre Q. The diagram QRHQTS thus obtained is of the form of a distorted figure 8.

Thus, in the position PCDQ of the mechanism shown, produce DC to cut PQ in M. Measure MP and MQ, and calculate the ratio $\frac{MP}{MQ}$. Multiply this ratio by 10 (turns per minute), and plot the result to some scale along QD from Q, such as QR.

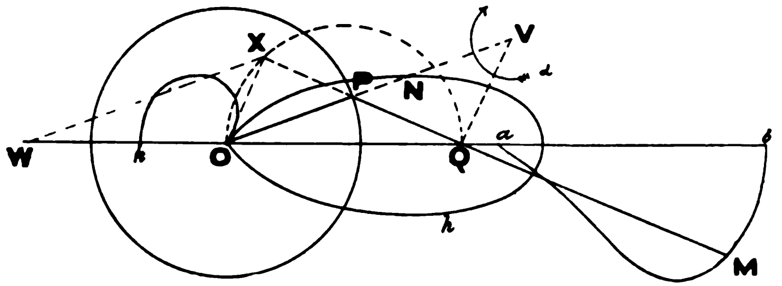
On the return movement of QD, when the arm arrives at the position shown, its angular velocity is represented by QS.

To draw the diagram of angular velocity of the connecting-link DE, through D (upper figure) draw a line parallel to the arm EC, cutting the line of centres in N. It has been shown before that V = $\delta . AD$, and V = $\beta . BD$; therefore—

$$\delta = \beta . \frac{BD}{AD} = \beta . \frac{BN}{NC}$$

Measure off the lengths BN and CN, and calculate their ratio. Plot this ratio along the arm BD from B, or, in the lower figure, along PC from P. The given diagram is thus obtained, namely, PGJPWKU. The angular velocity β of the smaller crank BD is generally assumed constant.

K 7



7. **The line OP (= 2 in.) turns uniformly about O at the rate of one turn per second. The line PM always passes through the point Q. Draw a diagram representing the angular velocity of the line PM, and trace out the locus of M.**

Take a number of positions of OP, and draw in PM for every position, noting the position of M in each instance. On joining up these positions, we get the curve *b*M*a*. This is the locus for half a revolution of OP; the other half is symmetrical about the centre line OQ.

Draw QV perpendicular to PQ, and produce OP to cut this perpendicular in V. Then at the instant the whole △ VPQ turns about V, as in the last problem, with an angular velocity, say, a. Then, if v is the linear velocity of P, we have $v = a \,.\, PV$. Also because P turns about O with constant angular velocity ω, then $v = \omega \,.\, OP$. Dividing one equation by the other, we get—

$$a = \frac{\omega \,.\, OP}{PV}$$

But, from similar triangles, $\frac{OP}{PV} = \frac{XP}{PQ}$ when XO is drawn from O perpendicular to PQ. Through X draw XW parallel to OP, cutting OQ in W; then—

$$\frac{WO}{OQ} = \frac{XP}{PQ} = \frac{OP}{PV}$$

and therefore—

$$a = \omega . \frac{WO}{OQ}$$

As OQ is constant, WO is a measure of the angular velocity of PQ. Plot lengths WO along their respective radii, and the curves O*h* and O*k* are obtained. O*k* is only half of the full curve, the other part being symmetrical with the given half.

It will be noted that as OX is perpendicular to PQ, the locus of X is a semicircle on OQ as diameter.

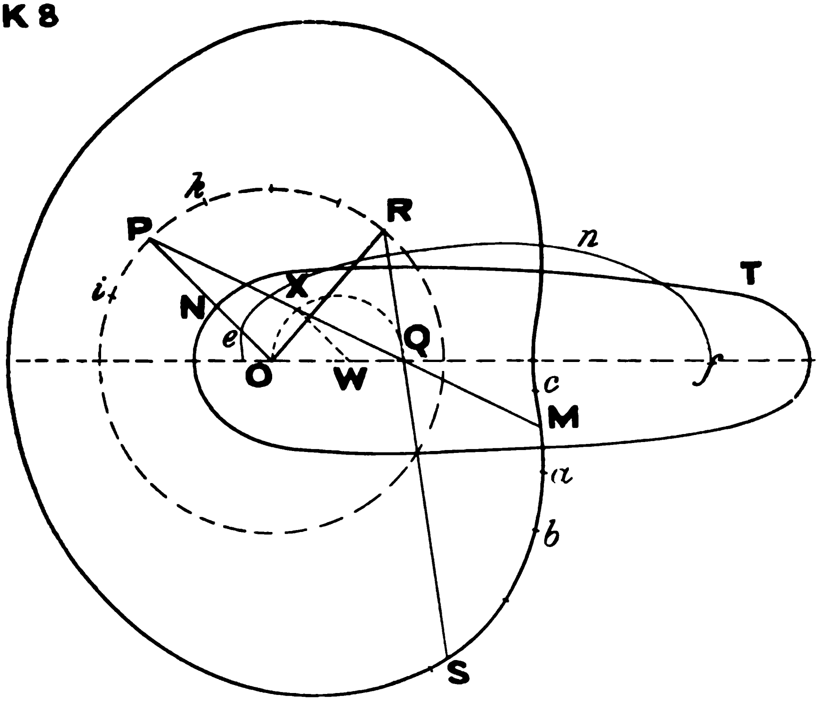
This diagram represents the mechanism of the oscillating cylinder engine and the pin-and-slit-bar quick-return shaping and slotting machines.

Fig. K. 8. **In the last problem the centre Q was situated at a distance of $3\frac{1}{2}$ in. from O. Now bring it up to within $1\frac{1}{2}$ in. of O, and trace the locus of M. Also draw a diagram of angular velocity of PQ, and a diagram of linear velocity of M.**

Proceeding as in the last example, describe a semicircle on OQ as diameter, and through the point X. Where PQM (or PM produced) cuts this semicircle draw a line, XW, parallel to OP, cutting OQ in W. Then OW represents the angular velocity of PQ. On OP plot ON = OW; then N is a point on the velocity curve. Repeat this construction for a great number of points, and the curve NT is obtained.

The locus of M is obtained by drawing the mechanism in a great number of positions. It is advisable to divide the circle R*k*P*i* into equal parts, such as *i*P, P*k*, etc., the corresponding positions of M being *c*, M, *a*, etc., when the end of OP is at *i*, P, *k*, etc.

K 8



The velocity of M has also been plotted along the radius vector OP, and is the curve *e*, *n*, *f*. Only half the curve is shown. As the distances *c*M, M*a*, etc., are moved over in equal times, they represent the velocity of M (see problems on velocity); hence plot half of CA along OP, such as O*e*. In the same way plot half of M*b* for the next position of OP.

A deck chair of the relative dimensions shown in the upper part of the accompanying figure sustains a load of 150 lbs., the whole of which comes on the sacking of the seat. It is required to find the direct stress in each member. The stresses due to bending and shearing need not be found.

The load of 150 lbs. will come to rest at the lowest possible point in the flexible seat, and as the length of same is constant, its locus is the ellipse shown, with foci at *a* and *b*. The lowest point is at *i*, where the horizontal tangent touches it. Set down *xz* to represent 150 lbs., and through the two ends of it draw lines parallel to *ai* and *bi*, meeting in *z*; then *xz* and *yz* represent the tensions in the sacking. To find the reactions at *e* and *f*, which are vertical, draw through *i* a vertical *in*, and another through *f*, whose length *fs* represents the load 150 lbs. Join *es*, then *mn* must represent the reaction at *f*; for if R is that reaction, then—

$$R \times fe = 150 \times me$$

or—

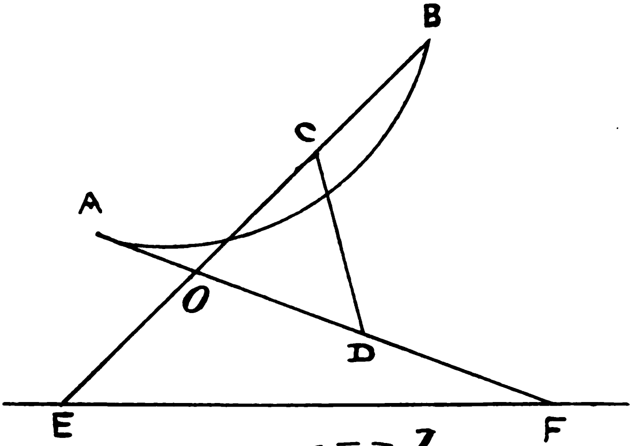
$$\frac{R}{150} = \frac{me}{fe} = \frac{mn}{fs}$$

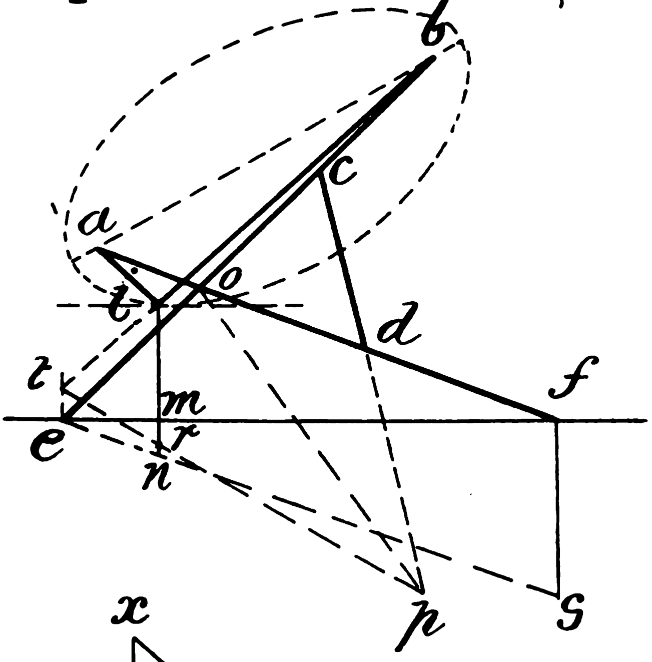
Further, 150 − R = reaction at *e*.

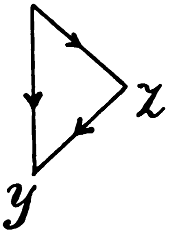
The piece *bcoe* is acted on by four forces, namely, the reaction (vertical) at *e*, the tension of the sacking at *b*, the thrust of *cd* at *c*, and the reaction of the pin at *o*, altogether unknown, except its point of application *o*. The resultant of the tension at *b* and the reaction at *e* is *wv*, where *wu* represents the reaction at *e*, and *uv* the tension at *b*. The actual directions of these two forces meet at *t*, therefore through *t* draw *tp* parallel to *vw*. The direction of thrust of *cd* must be along *cd*, intersecting *tp* in *p*; hence the reaction at *o* must pass through *p*.* Join *op*, and draw a triangle of forces whose sides are parallel to *pc*, *po*, and *pt*, of which the side parallel to *pt* equals *vw*. This triangle will give the thrust in *cd* and the reaction at *o*.

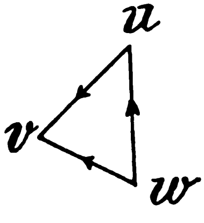
---

* See Pullen's "Graphic Statics," p. 12.


B
C
A
O
D
E
F


b
c
a
o
t
d
t
m
f
e
r
n
p
g

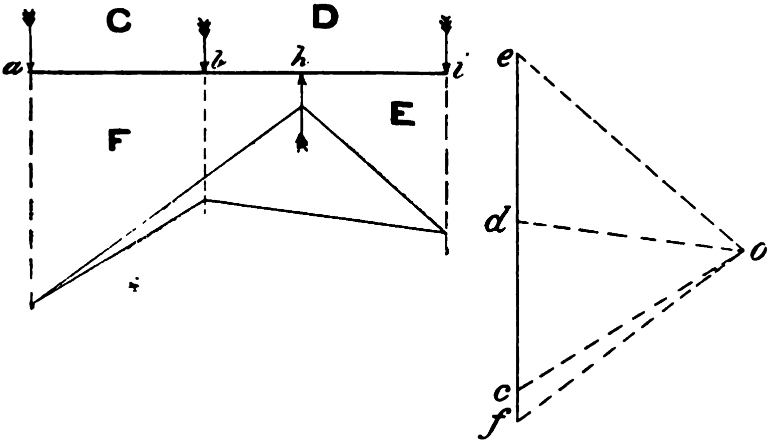

x
z
y


u
v
w

For question, see No. 65 of 1892.

Draw a horizontal line, *abhi*, of which *hi* represents $6\frac{3}{4}$ in., *bh* represents $4\frac{1}{4}$ in., and *ah* represents 12 in. Each of the forces at *b* and *i* are half the load of 150 lbs.

Set down the line of loads *edcf*, in which *ed* = *dc* = 75 lbs., and take any pole *o*. Draw the radial lines *oe*, *od*, and *oc*, and then the funicular polygon, with its closing line through the space F. Through *o* draw a line parallel to the closing line of the funicular polygon, cutting *ec* in *f*; then *cf* represents the required force to be applied at *a*.



INDEX.

Year.	No. of question.	Page.	Chap.
1887	41	15, 368	1, 9
"	42	52	1
"	43	32	1
"	44	148	4
"	45	138	4
"	46	122	3
"	47	116	3
"	48	219	5
"	49	174	5
"	50	169	5
"	51	282	7
"	52	262	6
"	53	312	8
"	54	80	2
"	55	62	2
1888	41	42	1
"	42	16	1
"	43	5	1
"	44	150	4
"	45	208	5
"	46	114	3
"	47	108	3
"	48	202	5
"	49	264	6
"	50	284	7
"	51	186	5
"	52	182	5
"	53	314	8
"	54	86	2
"	55	74	2
1889	41	14	1
"	42	4	1
"	43	18	1
"	44	142	4
"	45	146	4
"	46	110	3
"	47	140	4
"	48	112	3
"	49	192	5
"	50	188	5
1889	51	172	5
"	52	286	7
"	53	318	8
"	54	82	2
"	55	66	2
1890	51	26	1
"	52	36	1
"	53	49	1
"	54	136	4
"	55	144	4
"	56	126	3
"	57	156	4
"	58	160	4
"	59	196	5
"	60	194	5
"	61	190	5
"	62	294	7
"	63	320	8
"	64	92	2
"	65	90	2
1891	51	20	1
"	52	50	1
"	53	28	1
"	54	222	5
"	55	120	3
"	56	212	5
"	57	230	5
"	58	152	4
"	59	128	3
"	60	177	5
"	61	210	5
"	62	288	7
"	63	322	8
"	64	84	2
"	65	78	2
1892	51	44	1
"	52	54	1
"	53	56	1
"	54	134	4

Year.	No. of question.	Page.	Chap.
1892	55	132	4
,,	56	224	5
,,	57	180	5
,,	58	250	5
,,	59	184	5
,,	60	106	3
,,	61	290	7
,,	62	266	6
,,	63	326	8
,,	64	72	2
,,	65	60, 398	2, 9
1893	51	1, 2	1
,,	52	46, 48	1
,,	53	6, 8, 369	1, 9
,,	54	252	5
,,	55	204	5
,,	56	216	5
,,	57	227	5
,,	58	124	3
,,	59	214	5
,,	60	228	5
,,	61	292	7
,,	62	268	6
,,	63	328	8
,,	64	100	2
,,	65	70	2
1894	51	58	1
,,	52	24	1
,,	53	34	1
,,	54	162	4
,,	55	154	4
,,	56	226	5
1894	57	156	4
,,	58	118	3
,,	59	232	5
,,	60	240	5
,,	61	296	7
,,	62	334	8
,,	63	330	8
,,	64	68	2
,,	65	64	2
1895	51	30	1
,,	52	38	1
,,	53	130	3
,,	54	166	4
,,	55	103	3
,,	56	238	5
,,	57	234	5
,,	58	298	7
,,	59	332	8
,,	60	94	2
,,	61	76	2
1896	51	10, 12	1
,,	52	22	1
,,	53	164	4
,,	54	198	5
,,	55	244	5
,,	56	242	5
,,	57	248	5
,,	58	300	7
,,	59	336	8
,,	60	88	2
,,	61	97	2

PRINTED BY WILLIAM CLOWES AND SONS, LIMITED, LONDON AND BECCLES.